A STUDY OF COSMOLOGY WITH GRAVITATIONAL WAVES AND PRIMORDIAL BLACK HOLES

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by

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(2) Arnab Sarkar, Amna Ali and Salah Nasri ;
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(3) Arnab Sarkar, Amna Ali, K. Rajesh Nayak and A. S. Majumdar ;
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(4) Arnab Sarkar, Sabiruddin Molla and K. Rajesh Nayak ;
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অর্ণব সরকার ;

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পিএইচডি অধীক্ষক: অধ্যাপক অর্চন শুভ্র মজুমদার, আচার্য সত্যেন্দ্রনাথ বসু মৌলবিজ্ঞান কেন্দ্র , কলকাতা;

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এই গবেষণামূলক প্রবন্ধের প্রথম অধ্যায়ে মহাকর্ষীয় তরঙ্গ ও আদি কৃষ্ণগহ্বর সম্পর্কে কিছু মৌলিক বিষয় আলোচনা করা হয়েছে। দ্বিতীয় অধ্যায়ে ‘বাইনারি’ বা দ্বৈতগঠনে থাকা আদি কৃষ্ণগহ্বর সমূহের দ্বারা সৃষ্ট "স্টোকাস্টিক ব্যাকগ্রাউন্ড" অর্থাৎ সমগ্র মহাকর্ষীয় তরঙ্গ প্রেক্ষাপটের অনুসন্ধান করা হয়েছে ; যেখানে দ্বৈতগঠনে থাকা আদি কৃষ্ণগহ্বরগুলি পারিপার্শ্বিক অত্যন্ত বেশি ঘনত্বের বিকিরণ বা 'রেডিয়েশন' কে শোষণ করে চলেছিল। দ্বৈতগঠনে থাকা আদি কৃষ্ণগহ্বর গুলির ভরের অনবরত বৃদ্ধির জন্য, সেগুলি থেকে সৃষ্টি হওয়া মহাকর্ষীয় তরঙ্গের প্রশস্ততা-মানের যে সংশোধন বা পরিবর্তন-মূলক পদের উৎপত্তি হয়, আমরা প্রথমে সেগুলি গণনা করেছি। এরপর দেখানো হয়েছে যে এই সংশোধন বা পরিবর্তন গুলি দ্বৈতগঠনে থাকা আদি কৃষ্ণগহ্বর গুলি থেকে উৎপন্ন সমগ্র মহাকর্ষীয় তরঙ্গ প্রেক্ষাপটের ক্ষেত্রেই থাকে এবং কিছু কিছু ক্ষেত্রে এই সংশোধন বা পরিবর্তনগুলি তাৎপর্যপূর্ণ হতে পারে । এরপর বর্তমান ও ভবিষ্যৎ মহাকর্ষীয় তরঙ্গ শনাক্তকারী ব্যবস্থাগুলির সাহায্যে এই সমগ্র মহাকর্ষীয় তরঙ্গ প্রেক্ষাপটের শনাক্তকরণের সম্ভাবনা আলোচনা করা হয়েছে।

তৃতীয় অধ্যায়ে মহাকর্ষীয় বিকিরণ অর্থাৎ মেট্রিক এর (স্থান-কালের) তারতম্যের জন্য তড়িৎ-চুম্বকীয় স্ববল বা "সেল্ফ ফোর্সে" তৈরি হওয়া তারতম্যের বিশ্লেষণ করা হয়েছে। যে সমস্ত পরিস্থিতিতে এই তড়িৎ-চুম্বকীয় স্ববল এর তারতম্যের ফলে সৃষ্টি হওয়া পদগুলি মহাকর্ষীয় স্ববলের তুলনায় গুরুত্বপূর্ণ হয়ে ওঠে, সেই সব পরিস্থিতি ও শর্তগুলি খুঁজে বের করা হয়েছে ; এবং যেসব বিভিন্ন জ্যোতির্বিজ্ঞান সম্বন্ধীয় ও মহাজাগতিক ক্ষেত্রে এই পরিস্থিতি গুলি তৈরি হতে পারে তার আলোচনা করা হয়েছে।

চতুর্থ অধ্যায়ে একটি বিশেষ ধরনের "কে-এসেন্স ডার্ক এনার্জি"র (বর্তমান কালে মহাবিশ্বের বিস্তৃতি বা সম্প্রসারণের ত্বরনের জন্য দায়ী শক্তি বা কারণকে ‘ডার্ক এনার্জি’ নামে অভিহিত করা হয়) গোলাকার-প্রতিসম শোষণের জন্য হওয়া কৃষ্ণগহ্বরের ভরের পরিবর্তন এবং এর জন্য এই কৃষ্ণগহ্বর গুলি দ্বারা গঠিত দ্বৈতগঠন বা বাইনারিগুলির বিবর্তনের অনুসন্ধান করা হয়েছে। আমরা দেখিয়েছি যে এই ধরনের কৃষ্ণগহ্বর গুলির দ্বৈতগঠন থেকে বিকিরিত মহাকর্ষীয় তরঙ্গের গড় ক্ষমতা, অপরিবর্তিত ভরের কৃষ্ণগহ্বর দ্বারা গঠিত দ্বৈতগঠনগুলির তুলনায় অনেক বেশি দ্রুত বৃদ্ধি পায়। আবার এই ধরনের কৃষ্ণগহ্বর গুলির ভরের ক্রমাগত বৃদ্ধির ফলে, এগুলোর দ্বৈতগঠনের সমন্বিত বা একীভূত হওয়ার জন্য প্রয়োজনীয় সময়ের হ্রাসের পরিমাণও নির্ধারণ করা হয়েছে। এই কাজটি একই ধরনের অন্যান্য "স্কেলারফিল্ড ডার্ক এনার্জি"র উপস্থিতি ও তার শোষণের ফলে হওয়া কৃষ্ণগহ্বরের ভরের পরিবর্তনের জন্য সেই কৃষ্ণগহ্বর গুলি দ্বারা গঠিত দ্বৈতগঠনের বিবর্তন, তাদের একীভূত হওয়ার সময়ের পরিবর্তন এবং তাদের একীভূত হওয়ার হারের উপর এর সম্ভাব্য প্রভাব সূচিত করে।

পঞ্চম অধ্যায়ে একটি অঘূর্ণায়মান ও বৈদ্যুতিক আধানবিহীন আদি কৃষ্ণগহ্বরের স্থান-কাল, যাকে এক্ষেত্রে "জেনারেলাইজ ম্যাকভীটি মেট্রিক" হিসেবে নেওয়া হয়েছে, সেখানে অক্ষীয় তারতম্যের বিবর্তনের নির্ধারণকারী সমীকরণ নির্ণয় করা হয়েছে। এই সমীকরণটি থেকে এর বিভবকে ব্যবহার করে কিছু তাত্ত্বিক ব্যাখ্যা ও বিশ্লেষণ করা হয়েছে।

Abstract of the thesis :

In this thesis some cosmological phenomena associated with gravitational waves (GW) have been investigated. Though the main focus remains on gravitational waves associated with primordial black holes (PBHs), some other aspects have been explored too. PBHs are theoretically predicted to be created during the very early era of the Universe : starting from the end of inflation and approximately till the Big-bang nucleosynthesis, due to direct gravitational-collapse after 'horizon re-entry' in the regions with sufficiently deep density-fluctuations.
In the Introduction (chapter 1) some basic issues of GWs and PBHs, relevant to this thesis, have been discussed. In chapter 2, we have investigated the stochastic GW background produced by PBH-binaries during their early inspiral stage, while accreting high-density radiation surrounding those in the early Universe. We first calculate the correction terms appearing in the GW amplitude generated from such a PBH-binary due to changing PBH-masses. We show that the significance of the correction terms persists for the overall stochastic GW background produced from these PBH-binaries and discuss the detectability of this stochastic background.

In chapter 3, we have analyzed perturbation of electromagnetic self-force by metric-fluctuations viz. gravitational radiation. We derive the conditions of significance of the additional perturbative terms thus generated and discuss significance of the terms for various astrophysical and cosmological systems, including systems made up of PBHs in early Universe, where charged particles are in motion around the PBHs.
In chapter 4, the impact of change of masses of black holes, due to spherical accretion of k-essence dilatonic ghost-condensate model of dark energy on the evolution of the binaries formed with those, has been investigated. We find that the average power of the emitted GW from these binaries increases significantly faster than the constant-mass case. Furthermore, we estimate the reduction in coalescence time-intervals of the binaries due to the growth of the black hole masses. This work signifies the effect of accretion of similar scalar-field dark energies on the orbital evolution of binaries of black holes of certain mass-ranges, their coalescence time-scale and as a consequence their merging-rates too.

In chapter 5, we present the derivation of the equation governing the axial-perturbations in the space-time of a non-rotating uncharged PBH, produced in early Universe, whose metric is taken as the generalized McVittie metric. From this equation, we draw some physical interpretations using the potential.

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Chapter 1

Introduction :

Although a huge amount of theoretical work has already been done in studying astrophysics and cosmology with gravitational waves, there are still many aspects to explore. In this thesis we have investigated some cosmological phenomena associated with gravitational waves, which are unexplored till now. Though our main focus remains on gravitational waves associated with primordial black holes (PBHs), we have studied some other aspects too.
PBHs are supposed to be created during the very early era of the Universe (from the end of inflation to mainly till the Big-bang nucleosynthesis) due to direct gravitational-collapse in the regions with sufficiently deep density-fluctuations. The concept of PBHs was first proposed by Stephen Hawking in 1971. In this introductory chapter of the thesis, we first give a very brief review of the gravitational wave amplitude produced by binary of compact objects and the formulation of stochastic gravitational wave background produced from such sources, which are necessary for the works described in this thesis. Then, we discuss various facets of importance of PBHs in cosmology and gravitational wave astronomy.

1.1 Gravitational Waves : The new tool of observing the Universe

Using the analogy of the inverse square force law in electromagnetism and gravitation, the first possibility of existence of gravitational waves was discussed in 1893 by Heaviside. Subsequently, Poincare in 1905 proposed that gravitational waves propagate at the speed of light as a consequence of Lorentz invariance. In his general theory of relativity in 1916, Einstein formally predicted the existence of gravitational waves, which are waves in the geometry of space-time [1]. When any mass or system of masses is in a motion such that the system has at least a non-vanishing second-order time-derivative of the *Quadrupolar distribution*, then it produces ripples in the space-time geometry. These ripples are generally termed as waves because those are solutions to a wave equation.
From the time of theoretical proposal of their existence over a century ago, a large body of works has been performed to devise ingenious methods for their detection [2].
The first 'indirect' evidence of the existence of gravitational waves came from a binary consisting

of a neutron star and a pulsar [1] (known as PSR B1913+16, PSR J1915+1606 or PSR 1913+16). This was observed by R. Hulse and J. Taylor in 1974 [3]. The analysis of this binary system indicated that its orbital-period was changed in accordance with the prediction of general relativistic calculations, as is found for emission of gravitational waves from a binary of compact objects [4]. R. Hulse and J. Taylor received the 1993 Nobel prize for this discovery.

However, direct experimental detection of gravitational waves was not possible until 2015. The first direct detection of gravitational waves was reported in 2016 by LIGO scientific collaboration from a merging black hole binary [5], initiating a new era in observational astrophysics and cosmology. This was followed by detections from a few other binary sources in quick succession [6, 7, 8, 9, 10] (including one neutron star binary). R. Weiss, B. Barish and K. Thorne were awarded the 2017 Nobel prize in physics for their seminal contribution in this discovery. Gravitational wave astronomy opens up the possibility of exploring a host of issues relevant to fundamental physics and is of tremendous importance for specially those sources, which do not emit any observable electromagnetic signal.

The most widely studied sources of gravitational waves are the binaries of black holes [11]. Binaries of neutron stars and black hole-neutron star binaries are of similar importance too. Other mechanisms of production of gravitational radiation which have been discussed in various works till date, include nearby fly-pass of two compact objects [12], gravitational collapse of sufficiently massive stars [13], cosmological phase transitions [14], breaking of cosmic strings [15] and, inflation and preheating [16]. Gravitational wave observations can impose constraints on the theories of gravity and the early universe, as well as on extra-dimensional and braneworld models [17].

Here, we give a very brief overview of the basic parameters of gravitational wave produced by a binary of compact objects, specially those which are quite essential for this thesis. Under the *Quadrupole-approximation*, the amplitude of gravitational wave produced by any source can be given by [18]

$$h_{ij} = \frac{2G}{Dc^4}\ddot{\mathcal{I}}_{ij}\,, \tag{1.1}$$

where $\mathcal{I}_{ij}$ is the *'Quadrupole moment'* of the system and the 'dot's denote time-derivative. D is the spatial-distance from the source to the observer or detector.

Using the above formulae 1.1, the amplitude of gravitational wave produced by a binary of compact objects, situated at a cosmological distance, can be derived and this is given by [19, 20]

$$h_{ij}(t^{ret}_{obs}) = \frac{4}{r}\left(\frac{G\mathcal{M}}{c^2}\right)^{5/3}\left(\frac{(1+z)\pi f_{obs}(t^{ret}_{obs})}{c}\right)^{2/3} e_{ij}\,, \tag{1.2}$$

where $\mathcal{M}$ is called the *'Chirp-mass'* of the binary and for a binary consisting of compact objects of masses m_1 and m_2, it is defined as :

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}\,. \tag{1.3}$$

In the expression of $h_{ij}(t^{ret}_{obs})$ given in 1.2, f_{obs} is the observed frequency of the gravitational wave at the observer viz. the detector, t^{ret}_{obs} is the retarded time of observation of the gravitational wave signal, measured at the observer, z is the cosmological redshift at which the binary emitted the gravitational wave and r is the cosmological comoving distance, corresponding to the redshift z.

[1] A pulsar is also a neutron star, but highly magnetized and rotating. It emits beams of electromagnetic radiation from its poles

e_{ij} is the polarization-tensor.
The above expression 1.2 of $h_{ij}(t^{ret}_{obs})$ can also be expressed as :

$$h_{ij}(t^{ret}_{obs}) = \frac{4}{D_L(z)}\Big(\frac{G(1+z)\mathcal{M}}{c^2}\Big)^{5/3}\left(\frac{\pi f_{obs}(t^{ret}_{obs})}{c}\right)^{2/3} e_{ij}\,, \tag{1.4}$$

where $D_L(z)$ is the cosmological luminosity distance of the binary corresponding to the redshift z and its relation with r is $D_L = (1+z)r$, when the present scale-factor $a_0 = a(t_0)$ of the flat FLRW-Universe is set to 1.

The gravitational wave signatures can be broadly classified into two categories : (i) individual gravitational wave signals and (ii) stochastic gravitational wave backgrounds. A stochastic background is produced when waves produced from a random distribution of sources with all possible direction of propagation superimpose.
The overall gravitational wave amplitude h_{ij} of the stochastic background due to superimposition of many random amplitudes $h_{\mathcal{P}}(f,\hat{n},t)$, integrated over all possible frequencies f and all possible directions $\hat{n}$, is given by :

$$h_{ij}(t,\vec{r}) = \sum_{\mathcal{P}=+,\times}\int_f df\int_{\hat{n}} d^2\hat{n}\, h_{\mathcal{P}}(f,\hat{n},t)\, e^{\mathcal{P}}_{ij}(\hat{n}) exp[-2\pi i f(t-\hat{n}.\vec{r}/c)]\,, \tag{1.5}$$

where $\mathcal{P}$ denotes the polarization associated with the amplitude $h_{\mathcal{P}}(f,\hat{n},t)$ and as it is clear, here the summation over two types of polarizations viz. plus(+) and cross($\times$) has been taken.
For calculating the parameters associated with the stochastic background, the standard formalism [20] is usually employed, assuming that the background is stationary, Gaussian, isotropic and unpolarized. Under these assumptions, the spectral density of the stochastic background $S_h(f)$ is defined as :

$$S_h(f) = \frac{1}{4}\frac{d}{df}\langle h_{ij}(t)h^{ij}(t)\rangle\,, \tag{1.6}$$

where $h_{ij}(t) \equiv h_{ij}(t,\vec{r}=0)$. The brackets $\langle\ \rangle$ over the scalar product $h_{ij}(t)h^{ij}(t)$ in this case denote the average taken over certain interval of time [20].
The advantage of describing the theory in terms of the spectral density $S_h(f)$ is that it is directly comparable with the noise in a detector, denoted by $S_n(f)$. The response to any stochastic gravitational wave background by the detector is given by

$$h(t) = \left(\frac{F}{4}\langle h_{11}h^{11} + h_{12}h^{12} + h_{21}h^{21} + h_{22}h^{22}\rangle\right)^{1/2}$$

Or,

$$h(t) = \left(\frac{F}{4}\langle h_{11}h^{11} + 2h_{12}h^{12} + h_{22}h^{22}\rangle\right)^{1/2}, \tag{1.7}$$

when $h_{12} = h_{21}$. In terms of the spectral density $S_h(f)$, $h(t)$ is given by :

$$h(t) = \langle h^2(t)\rangle^{1/2} = \Big(F\int_f df\, S_h(f)\Big)^{1/2} = \Big(\frac{F}{4}\langle h_{ij}(t)h^{ij}(t)\rangle\Big)^{1/2}, \tag{1.8}$$

where F is the angular efficiency factor, which for interferometric detectors is $F = 2/5$, and for cylindrical bar detectors $F = 8/15$ [20].

1.2 Primordial Black Holes : Some Basic Aspects

Primordial Black Holes (PBHs) are supposed to be created during the very early era of the Universe, from the end of inflation and approximately till the Big-bang nucleosynthesis, due to direct gravitational-collapse in the regions with sufficiently deep density-fluctuations after *horizon re-entry*. The concept of PBHs was first proposed by Stephen Hawking in 1971 [21]. There are proposals of several mechanisms of production of inhomogeneities, capable of producing PBHs after cosmic-inflation. PBHs might have affected the evolution of the Universe through various ways e.g. accretion of the surrounding dense radiation in the early Universe, Hawking evaporation, formation of binaries and consequent emission of gravitational waves etc. and many more. Moreover, PBHs within certain mass-ranges are thought to survive till late Universe and are potential candidates for explaining dark matter. Using the physical processes involving PBHs, certain observational constraints have been put on the abundance of PBHs in particular mass ranges. But, even after imposition of the stringent observational constraints, the existence of the PBHs within a narrow mass range, in the present Universe, is debated.
In this section, some basic aspects of the PBHs have been discussed.

1.2.1 Some physical quantities associated with primordial black holes :

The most accepted mechanism of production of PBHs is the direct gravitational collapse of sufficiently large density-fluctuations after *horizon re-entry*, at the end of inflation. Inflation could generate fluctuations of very large wave-length, even greater than the Hubble-horizon at that time i.e. $L >> cH^{-1}$, where L is the wave-length and H is the Hubble-parameter. After the end of inflation, the Universe started a phase of decelerated expansion and in this phase, the large density-fluctuations would enter into the Hubble-horizon. This phenomenon is called 'Horizon re-entry'. After the horizon re-entry, if the amplitude of a density-fluctuation was sufficiently large, then that would lead to 'Jeans instability' and gravitational collapse of the density-fluctuation would have started. Consequently a black hole would be produced from the gravitational collapse. In the early Universe, the PBHs produced at time t (in seconds) after Big-bang, are predicted to have masses of the order of the particle horizon mass at the time of their formation, given by [22]

$$m_H(t) \approx \frac{c^3 t}{G} \approx 10^{15} \frac{t}{10^{-23}}\, g = 10^{38} t\, g. \tag{1.9}$$

[2] Hence, PBHs may span an enormous mass range from those produced at the end of inflation ($\sim\ 10^{-32}$ s) up to those produced at the Big-bang nucleosynthesis ($\sim$ 1 s) [23]. Till the time t$\approx\ 10^{-25}$ s, almost all PBHs had mass within the range ($<\ 10^{13}$ g) such that the Hawking-evaporation was dominant over the accretion of surrounding radiation for them, leading to decrease in their masses. On the otherhand, PBHs produced after time $> 10^{-25}$ s would have mass gain by accretion of the surrounding highly dense radiation, which dominates over the mass-loss due to Hawking evaporation.

It is pertinent to mention that B. J. Carr did a pioneering analysis of the mass-function of PBHs

[2] 'g' stands for *grams*.

in his work in 1975 [24]. According to B. J. Carr [24, 25], if the primordial fluctuations obey a Gaussian distribution, the probability, that a collapsing spherical region of initial mass m has a density contrast in the range δ to $\delta + d\delta$, is given by [26]

$$P(\delta, m)d\delta = \frac{1}{\sqrt{2\pi}\sigma(m)} exp.\Big(- \frac{\delta^2}{2\sigma(m)^2}\Big) \Theta\Big[\alpha\Big(\frac{m}{m_0}\Big)^{2/3} - \delta\Big] d\delta \,, \tag{1.10}$$

where $\sigma(m)$ is the mass variance and Θ denotes the 'Heaviside function', which represents the fact that the distribution is cut off above a maximum value of the density contrast $\delta_{\rm max}$. α is a constant of order 1.

The cumulative density of PBHs at time t is given by [24]

$$\rho_{PBH} = \int_{m_{min}}^{m_{max}} \Pi(m')\rho\, dm' \,. \tag{1.11}$$

where ρ is the background radiation density and $\Pi(m)dm$ is a quantity related to the probability that a spherical region having initial mass between m to $m + dm$ collapses to a PBH, which ultimately remains a single black hole, i.e., is not engulfed by any larger black hole [24]. The integral in the equation 1.11 has been performed from a minimum mass m_{min} to a maximum mass m_{max}. The mass-variance $\sigma(m)$ is taken as $\sigma(m) = \epsilon(m/m_0)^{-n}$, where ϵ is amplitude of mass-variance of primordial fluctuations, m_0 is the initial sub-horizon mass and $n = 2/3$ [24]. Hence, $\Pi(m)$ can be obtained as

$$\Pi(m) \sim \frac{1}{m}\epsilon\Big[exp\Big(- \frac{\mathscr{B}^4}{2\epsilon^2}\Big)\Big] \,, \tag{1.12}$$

where, $\mathscr{B}^2 \sim w$, with w being the equation of state parameter of the concerned cosmic-fluid, which is radiation in this case.

1.2.2 Observational constraints on primordial black holes :

It is worthwhile to mention certain observational constraints on the abundance of PBHs in particular mass ranges [27, 28, 29]. From the absence of noticeable microlensing, the ERS and MACHO surveys have excluded large abundances of PBHs in the mass-range 10^{26} to 10^{34} g [30, 31, 32]. This constraint can be indirectly applied to the abundance of PBHs of masses $< 10^{26}$ g in the early Universe. However, setting these mass limits are highly model-dependent [33] and regrouping of PBHs in dense halos can evade the microlensing constraints. The absence of some characteristic spectral distortions of the Cosmic Microwave Background's spectrum imposes constraints on the PBH abundance in the early Universe. Planck observations exclude PBHs of the mass-order $10^{35} - 10^{37}$ g from being a significant fraction of dark-matter [34]. However, the distortion constraints are results of complex processes and subject to considerable uncertainties. Moreover, it has been argued [35] that the rate of binary formation and consequent merging of PBHs in the early Universe could be significantly higher such that the PBHs produced with sub-stellar masses, bypassing the CMB-distortion constraints, would have grown by several orders of mass by the time of star formation. Hence, such PBHs could evade the most stringent microlensing constraints, as well.

1.2.3 Primordial black holes as a component of dark matter :

PBHs can be a good candidate for a fraction of dark matter, provided some of the PBHs survive till the present era of the Universe. An estimate says the PBHs should have a mass $\sim 10^{-19} M_{\odot}$, to have an age longer than the present age of the Universe [36, 37]. There are both advantages and challenges for PBH-dark matter models. One of the major advantages of PBH-dark matter models is that neither it requires any physics beyond the standard model of particle physics, which is generally required for modelling dark matter as any new or unknown kind of particles, e.g. axions ; nor does it need any sort of modification to the gravity beyond general theory of relativity.
Other advantages to be mentioned : the merging rate of binaries of black holes, inferred by the gravitational wave observations by aLIGO and VIRGO, can be obtained for two simple PBH-dark matter models. In one of those, dark matter halo mass function is extrapolated towards small scales [38] and in another, PBHs are regrouped in dense sub-halos as ultra-faint dwarf galaxies [39]. Furthermore, most of the PBH-dark matter models can solve the 'small-scale crisis' occurring in dark matter models with modified gravity theories and *Weakly Interacting Massive Particles (WIMP)* theories. For example, even a small population of PBHs of the order of solar mass, can heat the galaxy core, thereby solving the cusp-core problem [40, 41].
But, despite these upper-hands in favour of the PBH-dark matter models, there are some challenges for it too. These challenges include the fact that massive PBH populations could have induced certain signatures in the *Cosmic Microwave Background (CMB) anisotropy spectrum* [42, 43] and could have been detectable by *microlensing events* of electromagnetic signals emitted from stars in the Magellanic clouds. However, it has been proposed in some works that the signatures of this kind of PBHs in the CMB-anisotropy spectrum are subjected to large uncertainties. Also, the microlensing constraints, from experiments like EROS, MACHOS and Kepler, can be naturally evaded if the PBHs are clustered in the galactic halos, so that the probability of finding such a cluster, in the line-of-sight of the Magellanic clouds being observed, is actually very low.
Again, some kind of constraints are applicable in case of coexistence of PBH-dark matter and particle dark matter models like WIMPs. In case of their coexistence, there should be a copious particle dark matter annihilation in halos accumulated around individual PBHs in the local Universe, which would eventually lead to detectable gamma-ray signals, thereby constraining the coexistence scenario strongly.
Therefore, further investigations, in both theoretical and observational aspects in this regards, are necessary to have a concrete insight in this topic and in the present age of gravitational wave astronomy, it is even more demanding.

1.2.4 Gravitational wave signatures from primordial black holes :

One of the most interesting prospects of possible ways of detection of PBHs is that they can produce characteristic gravitational wave signals in various ways. PBHs could form binaries in both early and late Universe, and it is one prominent way of generation of gravitational waves from PBHs. The rate of formation of binaries of PBHs and rate of their merging differ in early and late Universe. For the early Universe, stochastic background of gravitational waves is expected instead of individual signals from PBH-binaries. Various works have been done till date, studying the stochastic gravitational wave background produced from PBH-binaries and their merging in

the early Universe [44, 45].
In case of late Universe, if some PBHs within certain mass ranges survive till the present era of the Universe, it is expected that those are clustered in galactic-halos and may make a fraction of dark matter, as has been described in the previous subsection 1.2.3. Then some of these PBHs can form binaries.
If we look at the estimated masses of the black holes of merging binaries from the catalog of sources of gravitational wave observations by LIGO scientific collaboration and VIRGO collaboration [46, 47], then we see that most of those black holes are more massive than the stellar mass black holes i.e. astrophysical black holes, which are usually observed by the techniques based on electromagnetic signals. This fact triggered a debate, even just after the announcement of the first direct detection of the binary-merger event by the LIGO scientific collaboration, that those black holes were not astrophysical black holes and might be of primordial origin [48, 38].
Besides the binary formations of PBHs, there are many other ways in which gravitational waves could be produced from them. These include vibrations or perturbations of PBHs. Perturbation of PBHs, leading to the emission of gravitational radiation in the forms of *quasi-normal modes* i.e. characteristic modes of vibration, can be produced by different types of phenomena such as : accretion of matter (viz. dense radiation in the early Universe) by PBHs, the newly born black holes in the ringdown stage after merging of two PBHs in binary formation, falling of gravitational waves from other sources on PBHs etc..

1.2.5 Hawking evaporation of primordial black holes :

The fact that some of the PBHs born in the early Universe, might have very small masses, starting from $\sim 10^{-5}\,g$ for those born just after the end of inflation, tempted Stephen Hawking to study their quantum properties and this ended up in the famous theoretical invention of quantum evaporation of black holes [49], named as 'Hawking evaporation'. Due to Hawking evaporation, black holes with mass m radiate with a temperature $T \approx 10^{-7}(m/M_\odot)^{-1}\,K$, and completely evaporate in a time $\tau \approx 10^{55}(m/M_\odot)^3\,Gy$.
Hawking evaporation has many facets of significance for black holes in cosmology. Before mentioning others, the issue, which is most apposite for this thesis, is the decrease in mass of PBHs due to Hawking evaporation. The time-rate of mass-loss for a black hole of mass m due to Hawking evaporation can be expressed as [23]

$$\frac{dm}{dt} = 5.34 \times 10^{-5} f(m) \left(\frac{m}{10^{10}\,g}\right)^{-2} \quad s^{-1} \quad , \tag{1.13}$$

where $f(m)$ is a measure of the number of emitted particle species, normalised to unity for a black hole with $m >> 10^{17}\,g$, emitting only particles which are (effectively) massless: photons, neutrinos and gravitons. Hence, lesser the mass of a PBH, more will be the time-rate of loss of mass. As a result, for PBHs within a certain range of mass, the rate of mass-loss due to Hawking evaporation is expected to be so immense, that the spherical accretion of the surrounding high-density radiation in that early era of Universe would be insufficient to make growth in their masses. An approximate estimation implies that till the time $t \approx 10^{-25}$ s, almost all PBHs have mass within the range ($< 10^{13}$ g) such that the Hawking evaporation is dominant over the accretion of radiation for those, leading to decrease in their masses. PBHs produced after

$t \approx 10^{-25}$ s undergo mass gain by accretion of the highly dense radiation, that dominates over the mass-loss due to Hawking evaporation. Thus Hawking evaporation plays an important role in change of masses of the PBHs within the specified mass-ranges.
The PBHs with mass $\sim 10^{15}$ g would have an evaporation time-scale of the order of the present age of Universe and hence these are expected to be exploding around the present time [23]. During their last stage of complete evaporation, they would produce photons of energy ~ 100 MeV. This fact has been used to impose observational constraints on their abundance, from the γ-ray background intensity.

1.2.6 Primordial black holes as seeds of supermassive black holes :

There are plenty of observational evidences that quasars powered by supermassive black holes (SMBHs) in the mass range $\sim 10^{8} - 10^{10} M_{\odot}$, existed at redshift z $\geq$ 6 [50, 51, 52, 53] . This has been a challenge for many theories of galaxy formation. Even if continuous accretion and successive merging is considered, it is very difficult for astrophysical black holes to produce those SMBHs at that redshifts. To solve this problem, some cosmologists proposed theories where PBHs within certain mass ranges might be responsible for the birth of those SMBHs. There are various possible scenarios where PBHs might give birth to those SMBHs.
First of all, some of the PBHs might themselves be of supermassive category. This is because it is well accepted that formation of PBHs from direct gravitational-collapse of sufficiently deep density fluctuations would have taken place from the end of inflation at $\sim 10^{-32}\,s$ age of the Universe, till the Big-bang nucleosynthesis at approximately $\sim 1\,s$ age of the Universe, as has been already stated. Hence, the PBHs produced during the Big-bang nucleosynthesis would have masses $\sim 10^{38}\,g$ i.e. $10^{5} M_{\odot}$. Again, these could grow further by accretion of surrounding matter and successive merging of the binary formations, to produce more massive black holes. Thus, this class of PBHs, themselves being SMBHs, could have helped in formation of galaxies through the 'seed' or 'Poisson effect' proposed in the 'seed theories' of galaxy formation [54]. Another possible way to produce SMBHs at the specified redshift, is by intermediate mass black hole (IMBH) seeds, having mass $\sim 10^{3} M_{\odot}$ at redshift z $\approx$ 15, which through uninterrupted accretion at the 'Eddington-limit' would produce the SMBHs. In 2004, N. Düchting proposed and studied the possibility whether these IMBH-seeds could be PBHs [55]. It is to be noted that PBHs produced at the age of Universe $\sim 10^{-2}$ s, would have masses $\sim 10^{3} M_{\odot}$.
Besides the above two procedures, Rachel Bean and João Magueijo proposed that PBHs would have produced those SMBHs by accretion of quintessence [56], which is a scalar-field model of dark energy.

1.2.7 Accretion by primordial black holes :

Besides Hawking evaporation, the other important means of changing of masses of PBHs is accretion of surrounding cosmic fluid. In both early and late Universe, PBHs would be accreting the ambient cosmic fluid. However, the nature of accretion, the accreted cosmic fluid, the change of mass of a PBH due to accretion and other consequences of accretion vary among different epochs of the Universe. Here a brief overview of the change of masses of PBHs and some other important effects due to accretion of cosmic fluids at various eras of the Universe will be discussed.

First of all, in the very early era of the Universe, the Universe was dominated by radiation and the radiation density was robust. At that epoch, it is expected that PBHs would undergo mass-gain by spherical accretion of the surrounding high-density radiation. In this regard, it should be mentioned that in this scenario, formation of accretion-disk by the radiation around PBHs was hardly possible, as the high-density radiation at that early era would not have sufficient angular-momentum to form an accretion-disk. [3] Therefore, we shall consider spherical accretion in case of accretion by PBHs in the early radiation dominated Universe.

The time-rate of change of mass of any black hole due to spherically accreting the ambient cosmic fluid has been an important theoretical topic of research from the onset of general theory of relativity and black hole physics. Among various works till date, we use the well-known result by E. Babichev et al. (2004) [57], which states that the time-rate of change of mass of a Schwarzschild black hole due to spherically accreting the surrounding cosmic-fluid, while the black hole is comoving with the cosmic fluid, is given by :

$$\frac{dm}{dt} = 4\pi\mathcal{A}\Big(\frac{Gm}{c^2}\Big)^2(1+w)\rho\,, \tag{1.14}$$

where w is the equation-of-state parameter and ρ is the background energy-density[4] converted to mass-density of the cosmic-fluid getting accreted, which is considered as a perfect-fluid. The constant $\mathcal{A}$ determines the energy-flux going into the black hole. The above time-rate of change of mass given in equation 1.14, can also be expressed as :

$$\frac{dm}{dt} = \mathcal{A}'(4\pi r_s^2)(1+w)\rho\,, \tag{1.15}$$

where r_s is the Schwarzschild-radius of the black hole i.e. $r_s = 2Gm/c^2$. In this form, it can be seen that the time-rate of change of mass of the black hole due to spherical accretion is directly proportional to the spherical-surface area with Schwarzschild radius, background density of the cosmic-fluid and the parameter $(1+w)$.

The detailed estimation of the numerical values of the constant $\mathcal{A}$ for various cases has been discussed in the work by E. Babichev et al. (2004) [57]. For the stationary spherical accretion of radiation in the early Universe, its value can be taken $\mathcal{A}' \sim c$ or ~ 1 in natural units (whence $c = 1$).

However, years before the work by E. Babichev et al., L. I. Petrich et al. (1989) [58] studied the time-rate of change of mass of a black hole due to spherically accreting the surrounding fluid, where the black hole is moving with some speed through the fluid i.e. the black hole and the fluid are non-comoving to each other. They derived the time-rate of change of mass of the black hole to be :

$$\frac{dm}{dt} = 4\pi\tilde{\mathcal{A}}\frac{G^2m^2}{(v_{rel}^2 + c_s^2)^{3/2}}n_\infty m_B\,, \tag{1.16}$$

where n_∞ is the background number-density of the fluid-particles and m_B is the mass of the particles constituting the fluid. The constant $\tilde{\mathcal{A}}$ can be taken to be ~ 1 [58]. v_{rel} is the relative-speed of the black hole w.r.t. the ambient fluid or vice-versa and c_s is the sound speed of the fluid getting accreted by the black hole. So, we can replace $n_\infty m_B$ as the background density of the

[3] Accretion-disks are formed around compact objects, so that the stellar matter or radiation undergoing accretion can impart its angular-momentum to that compact object.

[4] background density means density of the fluid from theoretically infinite distance from the black hole

fluid ρ and write the formulae 1.16 as :

$$\frac{dm}{dt} = 4\pi\tilde{\mathcal{A}}\frac{G^2m^2}{(v_{rel}^2 + c_s^2)^{3/2}}\rho\,. \tag{1.17}$$

However, these formula 1.14 and 1.17 may not give a good description of the change of mass of PBHs due to accretion of baryonic matter, specially in the late Universe. The accretion of baryonic matter is more complicated as gas-dynamics in the vicinity of PBHs is associated with this. If the PBH is in a region of the Universe with average baryonic matter density and temperature, then a competition between Bondi and Eddington-limited accretion [59] sets in. The accretion rate found in this way gives an information about the lower bound of relevant PBH accretion activity. The formation and evolution of structure in the late Universe would give rise to various types of uncertainties in the accretion-model and corresponding change of mass of a PBH. Again, due to structure-formation, as the late time Universe can be classified into various over-dense and under-dense regions, we expect that the density of baryonic and dark-matter indicated by a homogeneous FLRW model of the Universe would not give correct rate of change of mass of PBHs due to accretion ; and instead the spatially averaged density specific to an over-dense region should be used.

Chapter 2

Stochastic gravitational wave background from accreting primordial black hole binaries during early inspiral stage

2.1 Introduction

The last couple of years have seen several detections of gravitational waves from binary black hole mergers since the first report by the LIGO and VIRGO scientific collaborations [5, 6, 7, 8, 9, 10]. Besides detection of these individual sources, the stochastic gravitational wave backgrounds generated from unresolvable cosmological and astrophysical sources have also aroused interest. Among various sources of cosmological stochastic gravitational wave backgrounds, primordial black hole binaries formed in the early Universe are of considerable importance. Primordial black holes (PBHs) are produced in the early Universe by direct gravitational collapse of the regions containing sufficiently high density fluctuations of relativistic matter or radiation. It has been argued in some works that PBHs could survive up to present times and form a significant constituent of dark matter [61, 62]. It has also been argued [48, 38, 63, 64] that PBHs comprise the black hole merger event GW150914, leading to the first direct detection of gravitational waves.

One of the main mechanisms of formation of PBHs is the density fluctuations originating from the quantum vacuum fluctuations during inflation [65]. After the end of the inflation, the Universe entered a phase of decelerated expansion resulting in the density fluctuations re-entering the Hubble horizon. For a sufficiently large amplitude of fluctuation, Jeans-instability was triggered leading to the fluctuation collapsing to a PBH [66, 67]. Further, massive PBHs could also be formed due to collapse of large curvature perturbations generated during hybrid inflation [33]. A significant fraction of PBHs could have formed binaries which emitted gravitational waves in the course of gradual shrinking and merger [68]. Various aspects of stochastic gravitational waves from PBHs have been studied [69, 70, 71, 72, 44, 73]. Most of these works are related to the PBHs that formed during the late Universe [e.g.-[35, 70]], while a few have discussed some early Universe effects [44, 74].

There are certain key differences between the rate of formation of binaries of black holes in the early and late Universe. In the early Universe, the rate of expansion of Universe was so rapid that it had a significant effect on the rate of PBH binary formation. Moreover, the density of the background radiation was robust leading to a considerably higher rate of accretion. It has been argued that accretion of surrounding radiation can override the effect of Hawking evaporation leading to the net growth and longer survival of PBHs [75, 76, 77, 78]. In fact, such a phenomenon could be more prominent if the very early universe undergoes a phase of string- or brane-affected modified expansion [79, 80, 81, 82], or modified geometry for compact objects [83, 84]. The consequent mass gain persists during the subsequent standard radiation dominated expansion, and further impacts the rate of binary formation [85]. It has been recently argued that gravitational radiation due to mass variation can substantially exceed that due to orbital motion [86].

In the present work we focus on such PBH binaries in the early universe. Our motivation is to explore the effects of background expansion as well as accretion of radiation on the PBH binary parameters leading to modification of the emitted gravitational wave spectrum. We investigate the consequent alteration of the stochastic gravitational wave background, which, if detected, would lead to a direct signal of physics in the early universe, and may arguably provide a proof of existence of PBHs, as well.
The organisation of the chapter is as follows. In the next section we present a brief overview of binary formation by PBHs in the early universe. In section 2.3, we discuss the formalism for calculating the amplitude of gravitational waves from accreting PBH binaries. The stochastic background produced by them is computed in Section 2.4 where we further discuss the detectability of the resultant spectral density by present and future gravitational wave detectors. We conclude with a summary of our analysis in section 2.5.

2.2 Binary formation by primordial black holes in the early Universe

In the early Universe the PBHs produced at time t (in seconds) after Big-bang, have masses of order of the particle horizon mass at their formation epoch, given by [22]

$$m_H(t) \approx \frac{c^3 t}{G} \approx 10^{15} \frac{t}{10^{-23}}\, g = 10^{38} t\, g. \tag{2.1}$$

Hence, PBHs may span an enormous mass range from the end of inflation (10^{-32} s) up to the Big-bang nucleosynthesis (~ 1 s) [23]. Till the time t$\approx$ 10^{-25} s, almost all PBHs have mass-range ($< 10^{13}$ g) such that the Hawking evaporation is dominant over the accretion of radiation for them, leading to decrease in their mass. PBHs produced after $> 10^{-25}$ s, undergo mass gain by accretion of the highly dense radiation which dominates over the Hawking evaporation.

It has been already stated in the subsection 1.2.1 of the first Chapter (Introduction) that according to previous works [24, 25] if the primordial fluctuations obey a Gaussian distribution, the probability, that a collapsing spherical region of initial mass m has a density contrast in the range

δ and $\delta + d\delta$, is given by [26]

$$P(\delta, m)d\delta = \frac{1}{\sqrt{2\pi}\sigma(m)} exp.\Big(- \frac{\delta^2}{2\sigma(m)^2}\Big)\Theta\Big[\alpha\Big(\frac{m}{m_0}\Big)^{2/3} - \delta\Big]d\delta\,, \tag{2.2}$$

where $\sigma(m)$ is the mass variance and Θ denotes the Heaviside function, which represents the fact that the distribution is cut off above a maximum value of the density contrast $\delta_{\max}$. The cumulative density of PBHs at time t is given by [24]

$$\rho_{PBH} = \int_{m_{min}}^{m_{max}} \Pi(m')\rho\, dm'\,. \tag{2.3}$$

where ρ is the radiation density and $\Pi(m)dm$ is a quantity related to the probability that a spherical region having initial mass between m to $m + dm$ collapses to a PBH, which ultimately remains a single black hole, i.e., is not engulfed by any larger black hole [24]. Since in the present chapter we are interested in those PBHs for which the mass gain due to accretion of radiation is dominant over mass loss due to Hawking evaporation, we set the limits of the above integral to be $m_{min} = 10^{13}$ g and $m_{max} = m_H$. The mass-variance $\sigma(m)$ is taken as $\sigma(m) = \epsilon(m/m_0)^{-n}$, where ϵ is amplitude of mass-variance of primordial fluctuations, m_0 is the initial sub-horizon mass and $n = 2/3$ [24]. Hence, $\Pi(m)$ can be obtained as

$$\Pi(m) \sim \frac{1}{m}\epsilon\Big[exp\Big(- \frac{\mathscr{B}^4}{2\epsilon^2}\Big)\Big]\,, \tag{2.4}$$

where, $\mathscr{B}^2 \sim w$, with w being the equation of state parameter of the concerned cosmic-fluid, which is radiation in this case.

Binary formation of PBHs in the early Universe typically proceeds due to the decoupling of a pair of PBHs from the background cosmic expansion, with a third nearby PBH providing a tidal force to prevent head-on collision [87, 88, 85]. The scale-factor at which a pair of PBHs decouple from the cosmic expansion is given by [44]

$$a_{dc} \approx a_{eq}\Big(\frac{r_{dc}}{\tilde{r}}\Big)^3\,, \tag{2.5}$$

where, r_{dc} is the co-moving separation between the two black holes, $\tilde{r}$ is given by

$$\tilde{r}^3 = \frac{3}{4\pi}\frac{M}{a_{eq}^3\rho_{eq}} \tag{2.6}$$

and $M = m_1 + m_2$ is the total mass of the two PBHs decoupling from cosmic-expansion. a_{eq} and ρ_{eq} are respectively the scale-factor and density of cosmic-fluid at the matter-radiation equality. As the concerned era is radiation dominated, the corresponding time is given by

$$t_{dc} = \mathscr{A}^{-2}a_{dc}^2 \approx \mathscr{A}^{-2}a_{eq}^2\Big(\frac{r_{dc}}{\tilde{r}}\Big)^6\,, \tag{2.7}$$

where $\mathscr{A}$ is a constant defined by $a = \mathscr{A}t^{1/2}$. Here, the co-moving length-scale $\tilde{r}$ comes from the condition of decoupling from cosmic expansion, which is roughly when the mean mass of the pair of PBHs overtakes the mass of the cosmic fluid contained in the sphere of radius equal to the separation of the pair, given by

$$\frac{M}{2} > \frac{4\pi}{3c^2}\rho R^3\,, \tag{2.8}$$

where R is the proper separation between the PBHs and ρ is the density of the cosmic-fluid i.e. radiation. As argued in the references [48] and [44], the length-scale $\tilde{r}$ is such that $r_{dc} < \tilde{r}$ and consequently $a_{dc} < a_{eq}$. The decoupling time must be greater than the time of production of both the PBHs, given by equation (2.1), viz. $t \approx \frac{G}{c^3}m$. Hence, the combination of masses m_1 and m_2 in the total mass M (on which t_{dc} depends) should not be such that one of them is very large and the other is very small making the time of formation of the larger PBH greater than t_{dc}.

In the early inspiral stage, the angular-frequency of the PBH-binaries (just after formation of the binary) is given by : $\omega = (G(m_1+m_2)/R_{dc}^3)^{1/2}$, where substituting the expression of proper separation between them : $R_{dc} \approx \tilde{r}\frac{a_{dc}^{4/3}}{a_{eq}^{1/3}}$, we can obtain :

$$\omega = \Big(\frac{4\pi G(m_1+m_2)a_{eq}^4\rho_{eq}}{3\mathscr{A}^4 t_{dc}^2(m_1+m_2)}\Big)^{1/2} = \Big(\frac{4\pi G a_{eq}^4\rho_{eq}}{3\mathscr{A}^4 t_{dc}^2}\Big)^{1/2} \tag{2.9}$$

The radial distance of a PBH-binary at a scale-factor 'a' is given by the usual formula for cosmological distance,

$$D(a) = \frac{c}{H_0}\int_a^1 \frac{da}{a^2(\Omega_{DE}+\Omega_M a^{-3})^{1/2}}\,, \tag{2.10}$$

where, Ω_{DE} and Ω_M are the fractional densities of dark energy and non-relativistic matter at present Universe and we have neglected the fractional density of radiation at present Universe Ω_R, as $\Omega_R << \Omega_{DE}, \Omega_M$. The distance D of a PBH-binary is dependent on the masses of the PBHs constituting the binary since the time of binary formation depends on the total mass of the two PBHs and also on the initial comoving-separation between the PBHs after forming the binary. Substituting the values of Ω_{DE} and Ω_M and performing the integration, one obtains the distance $D(m_1, m_2, r_{dc})$ to a specific PBH-binary, given by

$$D(a) \approx \frac{c}{H_0}\Big(-\xi_1 + \frac{\xi_2\,{}_2\mathcal{F}_1(\frac{1}{3},\frac{1}{2},\frac{4}{3},-\frac{\zeta}{a^3})}{a}\Big). \tag{2.11}$$

Here, ${}_2\mathcal{F}_1$ is the Hypergeometric function, and ξ_1 , ξ_2 and ζ are quantities whose numerical values depend on Ω_{DE} and Ω_M.

2.3 Gravitational wave amplitude from accreting primordial black hole binaries

The second mass-moment, or quadrupole-moment in the Transverse-Traceless (TT) gauge, of a binary of compact objects is given by $I_{ij} \equiv \int \rho x_i x_j = \mu xy$, for cross-polarization, where the orbital-plane of the binary is chosen to be the xy-plane with origin at their center-of-mass, μ is a function of the masses of the compact objects, and it is assumed for simplicity that the z-axis is along the line from the center of the binary to the observer. So, if μ is constant,

$\dot{I}_{ij} = \frac{d}{dt}\int d^3x(\rho(t,x)x_i x_j) = \frac{d}{dt}(\mu x(t)y(t))$, which gives $\ddot{I}_{ij} = \mu(\frac{d^2x}{dt^2}y + x\frac{d^2y}{dt^2}) + 2\mu(\frac{dx}{dt}\frac{dy}{dt})$. However, if μ varies with time, there would be two extra terms, i.e., $\ddot{I}_{ij} = \frac{d^2\mu}{dt^2}xy + \frac{d\mu}{dt}(\frac{dx}{dt}y + x\frac{dy}{dt}) + \mu(\frac{d^2x}{dt^2}y + x\frac{d^2y}{dt^2}) + 2\mu(\frac{dx}{dt}\frac{dy}{dt})$. Hence, the gravitational wave amplitude in cross($\times$)-polarization from a single PBH-binary of continuously changing PBH masses m_1 and m_2, in the early inspiral stage where the Keplarian-approximations are valid, will be given by

$$h_\times = \frac{2G}{D\,c^4}\ddot{I}_{xy} = \frac{G^{\frac{5}{3}}}{D(m_1,m_2,r_{dc})\,c^4}\Big[\frac{m_1m_2}{(m_1+m_2)^{1/3}}\{-4\omega^{\frac{2}{3}}sin(2\omega t)\} + \Big\{\frac{d^2}{dt^2}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\Big\}\omega^{-\frac{4}{3}}sin(2\omega t) + 2\Big\{\frac{d}{dt}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\Big\}\{2\omega^{-\frac{1}{3}}cos(2\omega t)\}\Big]. \tag{2.12}$$

(the terms generated due to time-variation of angular frequency are negligible here.) In the RHS of the equation (2.12) the first term is the usual one for binaries of constant mass black holes, while the rest two are present if the masses of the black holes in the binary change with time.

The time-rate of change of mass of any non-rotating PBH in early Universe, due to spherical accretion of the surrounding radiation is approximately given by,

$$\dot{m} = 4\pi\mathcal{A}\Big(\frac{Gm}{c^2}\Big)^2(1+w)\rho \tag{2.13}$$

where the constant $\mathcal{A}$ determines the energy flux going into the black hole. The choice of numerical value of $\mathcal{A}$ has been already discussed in the subsection 1.2.7 of the chapter 1. This time rate of change of mass of a PBH is also valid when it is in the early inspiral stage of a binary. In order to have an idea of the rate of mass gain during the radiation dominated era, we plot $\dot{m}$ versus time for a range of PBH formation masses m_H in Fig. 2.1. One sees that $\dot{m}$ can indeed take large values during the early radiation dominated era, but falls rapidly with time. This is due to the fall in background radiation density.
Next, using the Friedmann's equations of FLRW-cosmology $H^2 = \frac{8\pi G}{3c^2}\rho$ and the conservation equations of energy-momentum tensor of the cosmic fluid, *viz.* $\dot{\rho} = -3H(1+w)\rho$ and substituting in Eq.(2.13), after taking its derivative, we get

$$\ddot{m} = \frac{4\pi\mathcal{A}G^2}{c^4}(1+w)\Big[-3\Big(\frac{8\pi G}{3c^2}\Big)^{1/2}m^2(1+w)\rho^{3/2} + 2\Big(4\pi\mathcal{A}\Big(\frac{G}{c^2}\Big)^2(1+w)m^3\Big)\rho^2\Big]. \tag{2.14}$$

A plot of $-\ddot{m}$ versus time in Fig. 2.2 reveals that the nature of variation of $-\ddot{m}$ with time is quite similar to that of the variation of $\dot{m}$ with time. It starts from huge values during the early radiation dominated era, while falls rapidly with time. The reason, for $\ddot{m}$ having negative values, is clearly the fall of $\frac{dm}{dt}$ with time due to decreasing background radiation density.

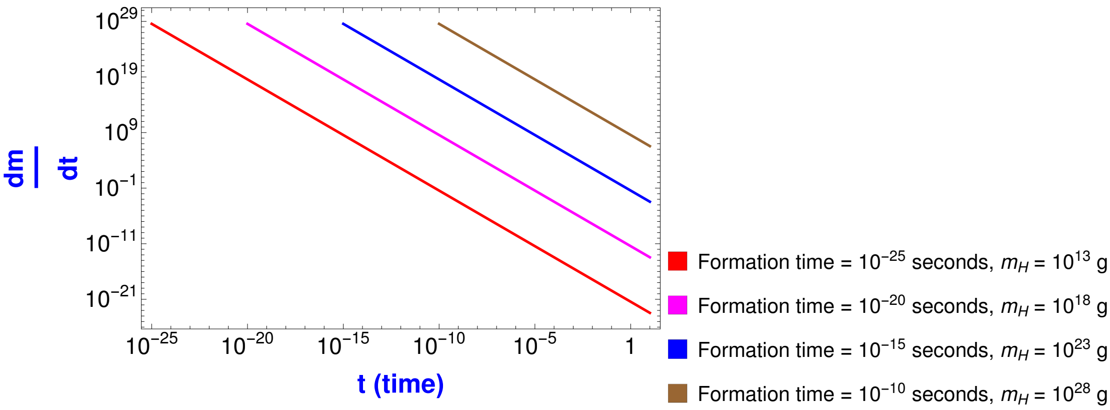

Figure 2.1: Plot of $\frac{dm}{dt}$ vs time t, where $\frac{dm}{dt}$ is in units of g/s and time t is in seconds after Big-bang. We have plotted a family of four curves for four different initial masses viz. m_H with the values 10^{28}, 10^{23}, 10^{18} and 10^{13} g.

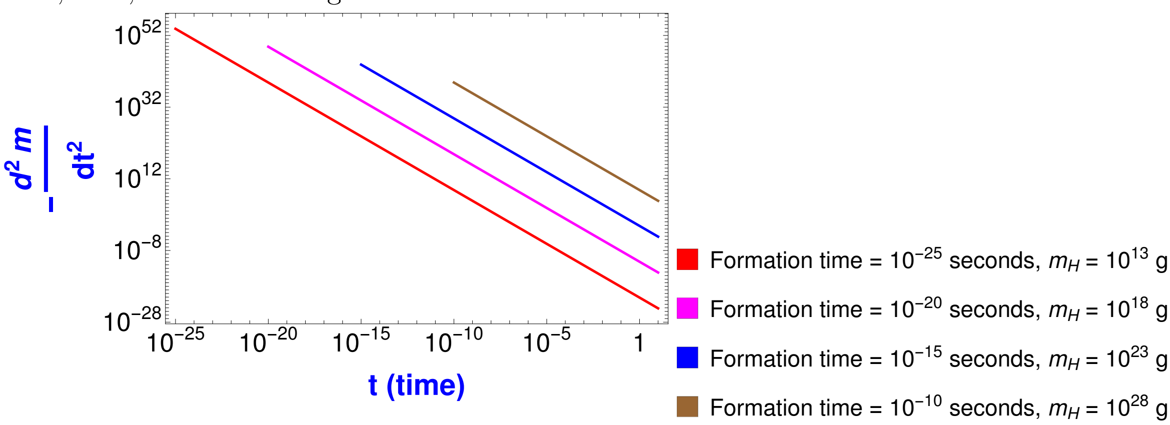

Figure 2.2: Plot of $-\frac{d^2m}{dt^2}$ vs time t, where $\frac{d^2m}{dt^2}$ is in units of g/s^2 and time t is in seconds after Big-bang. We have plotted a family of four curves for four different initial masses viz. m_H with the values 10^{28}, 10^{23}, 10^{18} and 10^{13} g.

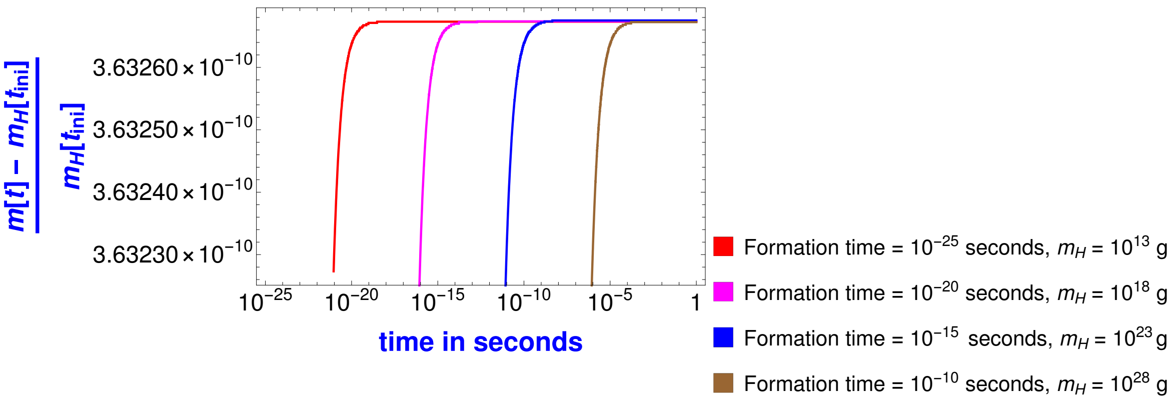

Figure 2.3: Plot of $\frac{m(t)-m_H}{m_H}$, which is the ratio of the growth in mass of a PBH, to its initial mass, w.r.t. time. The range of time shown in the figure is from 10^{-25} to 1 s after Big-bang. We have plotted a family of four curves for four different initial masses viz. m_H with the values 10^{28}, 10^{23}, 10^{18} and 10^{13} g.

Next, in Fig. 2.3 we give a plot of the time-variation of the ratio of change in mass of a PBH taken to its initial mass, with which it was born i.e. the horizon mass m_H. It is evident from this figure that the growth of the PBHs, for the specified range of initial masses, are negligible in comparison with their initial masses. The amount of growth of the PBHs' masses are less than of the order of 10^{-12} times of their initial masses, in the range of time of our interest. Hence, it is in clear agreement with the argument of B. J. Carr and S. W. Hawking in their work [89] that PBHs can not grow much significantly in the radiation dominated era. Various other works also suggest the same [90, 91]. It can also be noticed from this figure 2.3, that initially the masses grow faster for a little time, after which they tend to become constant. The reason behind this can be interpreted as the rapid fall in the background radiation density in this early radiation-dominated era, due to which the rate of growth of PBH masses also fall rapidly with the evolution of Universe. So, it is very interesting to note the fact that although the growth of masses of the PBHs are negligible when we compare that with their initial masses, but yet the rate of growth is sufficient to have a significant impact on the gravitational wave emitted from their binaries, which we shall show in our work.

Now, it is to be noted that both the $\dot{m}$ and $\ddot{m}$ have been expressed in terms of m and ρ, enabling one to write the single and double time derivatives of the chirp-mass function, in the correction terms in cross-polarization of gravitational wave amplitude in Eq.(2.12), respectively as :

$$\frac{d}{dt}\frac{m_1m_2}{(m_1+m_2)^{1/3}} = \left\{\frac{4\pi\mathcal{A}G^2}{c^4}(1+w)\rho\right\}\left[m_1m_2(m_1+m_2)^{2/3} - \frac{m_1m_2(m_1^2+m_2^2)}{3(m_1+m_2)^{4/3}}\right] \tag{2.15}$$

and

$$\begin{aligned} \frac{d^2}{dt^2}\frac{m_1m_2}{(m_1+m_2)^{1/3}} &= \frac{4\pi\mathcal{A}G^2}{c^4}(1+w)\rho\frac{m_1^2m_2^2}{(m_1+m_2)^{1/3}} + \\ \frac{4\pi\mathcal{A}G^2}{c^4}(1+w)^2\Big(&\frac{(\mathscr{C}_1m_1^2\rho^{3/2}+\mathscr{C}_2m_1^3\rho^2)m_2}{(m_1+m_2)^{1/3}} + \frac{(\mathscr{C}_1m_2^2\rho^{3/2}+\mathscr{C}_2m_2^3\rho^2)m_1}{(m_1+m_2)^{1/3}}\Big) \\ &-\Big(\frac{4\pi\mathcal{A}G^2}{c^4}(1+w)\rho\Big)^2\frac{m_1m_2(m_1+m_2)(m_1^2+m_2^2)}{(m_1+m_2)^{4/3}} \\ &-\frac{4\pi\mathcal{A}G^2}{c^4}(1+w)^2\frac{m_1m_2}{3(m_1+m_2)^{4/3}}(\mathscr{C}_1\rho^{3/2}(m_1^2+m_2^2)+ \\ &\mathscr{C}_2\rho^2(m_1^3+m_2^3)) + \frac{4\pi\mathcal{A}G^2}{c^4}(1+w)\frac{4m_1m_2(m_1^2+m_2^2)^2}{9(m_1+m_2)^{7/3}}, \end{aligned} \tag{2.16}$$

where the quantities $\mathscr{C}_1$ and $\mathscr{C}_2$ are, respectively, $-3(8\pi G/3c^2)^{1/2}$ and $8\pi\mathcal{A}(G/c^2)^2$. We can now calculate the numerical values of the peak magnitudes (without the sinusoidal variations) of the first and second corrections terms in gravitational wave amplitude given by $\frac{G^{5/3}}{D\,c^4}\left\{\frac{d^2}{dt^2}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\right\}\omega^{-\frac{4}{3}}$, and $\frac{G^{5/3}}{D\,c^4}2\left\{\frac{d}{dt}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\right\}\{2\omega^{-\frac{1}{3}}\}$ respectively, for any typical PBH binary and compare their values with that of the main term $\frac{G^{5/3}}{D\,c^4}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\{-4\omega^{\frac{2}{3}}\}$.

We plot these terms in Fig. 2.4 as functions of the black hole masses and the background radiation density, choosing $m_2 = 2m_1$ and separation between the PBHs is given by 100 times the sum of their Schwarzschild-radii (the angular frequency is to be directly obtained from Kepler's law as we are considering the early inspiral stage). We use the expression for cosmological distance in terms of the scale factor given by equation 2.11, considering the scale factor at which the PBHs constituting the binary were born (as masses of both the PBHs are of same order, their time of birth is also of approximately same order). It is evident from the plot that for certain cases the corrections are not only significant but also dominant. The constancy of the main term w.r.t. the background radiation density ρ can be clearly depicted in the plot below, as it is independent of ρ. With the increasing density of radiation ρ, both the correction terms increase. Therefore, the instantaneous rate of change of masses (both the single and double time-derivatives of the masses) of the PBHs in binaries have a significant effect on the gravitational wave amplitude generated by them and hence, on the overall stochastic gravitational wave background.

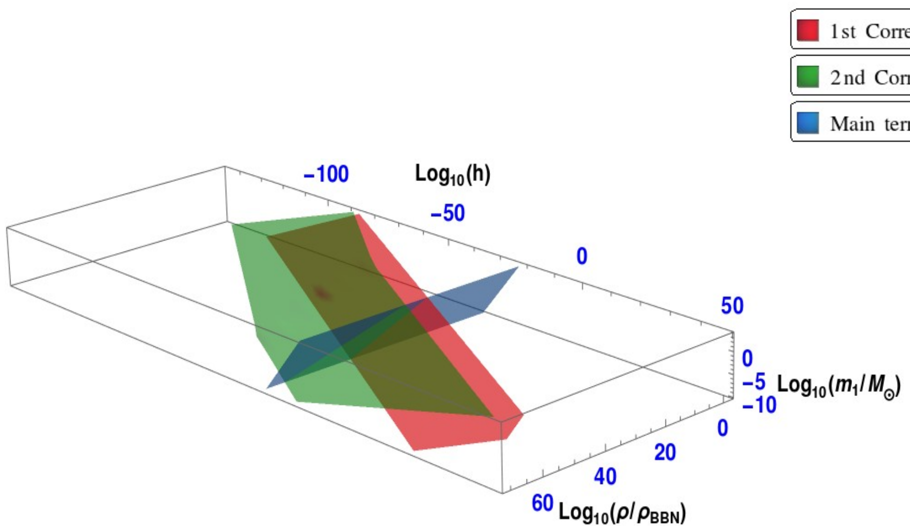

Figure 2.4: Variation of the three terms in the gravitational wave amplitude given by Eq.(2.12) produced by a PBH binary in the early inspiral stage, w.r.t. the background radiation density and mass, where $\rho_{BBN} \approx 10g/cm^3$ is the radiation density during big-bang nucleosynthesis, and $M_{\odot}$ is the Solar-mass.

2.4 Stochastic gravitational wave background and its detectability

Substituting the expressions of plus and cross polarized components of gravitational wave amplitude from an individual PBH-binary in the scalar product $h_{ij}h^{ij}$ one gets, [1]

$$\begin{aligned} h_{ij}h^{ij} = \frac{2\,G^{\frac{5}{3}}}{D(m_1,m_2,r_{dc})\,c^4}\Big[2\Big(2\frac{m_1m_2}{(m_1+m_2)^{1/3}}\omega^{\frac{2}{3}}\Big)^2 + 2\Big(2\Big\{\frac{d}{dt}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\Big\}\omega^{-\frac{1}{3}}\Big)^2 + \\ \Big(\Big\{\frac{d^2}{dt^2}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\Big\}\omega^{-\frac{4}{3}}\Big)^2 + 4\Big\{\frac{d^2}{dt^2}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\Big\}\omega^{-\frac{4}{3}}\Big(-2\frac{m_1m_2}{(m_1+m_2)^{1/3}}\omega^{\frac{2}{3}}\Big)\Big]\,. \end{aligned} \tag{2.17}$$

An additional cross-correction term appears due to the non-vanishing of the product between the main term and the second correction term, i.e. the term containing the double-time derivative of chirp-mass. Similar correction terms also appear in the response to the detector $h(t)$.
We have already discussed some basic parameters for a stochastic gravitational wave background in the section 1.1. Unlike the case of a single binary, h_{ij} for stochastic background of gravitational waves stands for the overall gravitational wave amplitude of the stochastic background, integrated over all possible frequencies and all directions, given by

$$h_{ij}(t,\vec{r}) = \sum_{\mathcal{P}=+,\times}\int_f df\int d^2\hat{n}\,h_{\mathcal{P}}(f,\hat{n},t)e^{\mathcal{P}}_{ij}(\hat{n})exp[-2\pi i f(t-\hat{n}.\vec{r}/c)]\,, \tag{2.18}$$

[1] The cross-polarized amplitude of gravitational wave from a PBH-binary of changing PBH masses is given in the equation 2.12 and the plus-polarized amplitude can also be obtained quite similarly using the basic formula.

where $h_{\mathcal{P}}$ is the gravitational wave amplitude produced from each PBH binary. The constituent gravitational waves from all the PBH-binaries come from all directions or the overall solid angle, as they are statistically distributed in the Universe and with statistically distributed parameters. Hence, for the stochastic background, one has to integrate over all the solid angles and over their masses.

Denoting the chirp masses as $\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$, the density of gravitational wave amplitude generated from PBH binaries in the differential chirp-mass range $\mathcal{M}$ to $\mathcal{M}+d\mathcal{M}$, for cross-polarization is given by

$$dh_{ij}(t,\vec{r}) = \int_f df \int d^2\hat{n}\, \mathcal{N}(\mathcal{M})d\mathcal{M}\, (h_\times e_{ij}^\times(\hat{n}))\,, \tag{2.19}$$

where $\mathcal{N}(\mathcal{M})d\mathcal{M}$ is the number-density of PBH-binaries in the differential chirp-mass range $\mathcal{M}$ to $\mathcal{M}+d\mathcal{M}$ at the concerned time. The total gravitational wave amplitude density generated for cross-polarization from all the PBH binaries in the chirp-mass range from $\mathcal{M}_{min}$ to $\mathcal{M}_{max}$ is given by

$$\int dh_{ij}(t,\vec{r}) = \int_f df \int d^2\hat{n} \int_{\mathcal{M}_{min}}^{\mathcal{M}_{max}} \mathcal{N}(\mathcal{M})d\mathcal{M}(h_\times e_{ij}^\times(\hat{n}))\,. \tag{2.20}$$

The formation of binaries is taken to proceed under the three-body configuration [44]. The differential co-moving number-density of PBH-binaries resulting from three-body configurations may be written as

$$d\mathcal{N}(r_1,r_2) = \frac{1}{2}(n(m_1)dm_1)(e^{-N(r_2)}dN(r_1,m_2)dN(r_2,m_3))\,. \tag{2.21}$$

The part $(e^{-N(r_2)}dN(r_1,m_2)dN(r_2,m_3))$ stands for the probability that those PBHs belong to the specified three-body configuration. The quantity $dN(r,m)$ is given by

$$dN(r,m) = 4\pi r^2 n(m)(1+\xi(r))drdm\,. \tag{2.22}$$

Here, $\xi(r)$ is the PBH two-point function [44]. In the simplest case, the two-point function can be taken as a constant $(1+\xi(r)) = \delta_{dc}$. The factor 1/2 in the RHS of equation 2.21 signifies the fact that the number of PBH-binaries would be just the half of the number of PBHs forming those binaries and $N(r_2) = \int dN(r_2,m)$ is the expected number of PBHs surrounding one PBH in the sphere of co-moving radius r_2. The quantity $N(r_2)$ is given by

$$N(r_2) = \int_0^{r_2}\int_{m_{min}}^{m} \delta_{dc}(4\pi r^2 dr)\Big(\rho\frac{\Pi(m)}{m}dm\Big)\,. \tag{2.23}$$

Substituting the expression of the distribution function given by Eq.(2.4), and performing the integrals over r and m in $N(r_2)$, one gets

$$N(r_2) = \delta_{dc}\rho\Big(\frac{4}{3}\pi r_2^3\Big)\epsilon\, exp.(\frac{-w^2}{2\epsilon^2})\Big(-\frac{1}{m}+\frac{1}{m_{min}}\Big)\,. \tag{2.24}$$

For brevity of notation we define $\mathscr{N}(m)$ as $N(r_2) = \frac{4}{3}\pi r_2^3 \mathscr{N}(m)$.

The total gravitational wave amplitude density generated (for cross-polarization) from all the

PBH-binaries is given by

$$h_{ij}(t,\vec{r}) = \int dh_{ij}(t,\vec{r}) = \int_f df \int d^2\hat{n} \int_{m_{1,min}}^{m_{1,max}} \int_{m_{2,min}}^{m_{2,max}} \int_{m_{3,min}}^{m_{3,max}} \int_{r_1=0}^{\tilde{r}} \int_{r_2=r_1}^{\infty} d\mathcal{N}(r_1,r_2)(h_\times e_{ij}^{\times}(\hat{n}))\,, \tag{2.25}$$

where $\tilde{r}$ is defined earlier in Eq.(2.6). Here, the integrations over r_1 and r_2 are respectively from 0 to $\tilde{r}$, and r_1 to ∞, because the second PBH should be within a radial distance 0 to $\tilde{r}$ from the first PBH, while the third PBH has to be anywhere outside r_1 ($r_2 > r_1$). The contribution of the main term (i.e. the term without derivatives of masses) to the gravitational wave amplitude density is :

$$h_{ij\,main}(t,\vec{r}) = \frac{1}{2}\rho^3\delta_{dc}^2 \int_f df \int_{\hat{n}} d^2\hat{n} \int_{m_{1,min}}^{m_{1,max}} \int_{m_{2,min}}^{m_{2,max}} \int_{m_{3,min}}^{m_{3,max}} \Big(\frac{\Pi(m_1)}{m_1}dm_1 \frac{\Pi(m_2)}{m_2}dm_2 \frac{\Pi(m_3)}{m_3}dm_3\Big) \left(4\pi \int_{r_2} r_2^2 e^{-N(r_2)}dr_2\right) 4\pi \int_{r_1} r_1^2 dr_1 \frac{G^{5/3}}{D(m_1,m_2,r_1)\,c^4} \frac{m_1 m_2}{(m_1+m_2)^{1/3}} (-4\omega^{2/3} sin(2\omega t))\,.$$

The contributions of the correction terms follow similarly. Note that the hypergeometric function contained in the expression of the distance $D(m_1,m_2,r_1)$ given by Eq.(2.11) can be written as

$${}_2\mathcal{F}_1(a,b,c,Z) \approx \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-Z)^{-a} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-Z)^{-b}\,, \tag{2.26}$$

for $|Z| >> 1$. This allows us to carry out the radial integrations analytically. Employing the approximation of the Hypergeometric function as described above, we find that the expession of the distance can be approximately written as :

$$D(m_1,m_2,r_1) \approx \mathscr{D}(1+\alpha\, a_{dc}^{1/2})^{-1}\,, \tag{2.27}$$

where numerical values of $\mathscr{D}$ and α are estimated to be of order ~ 1.
The contribution from the main term in $\langle h(t)^2\rangle$ is given by

$$\langle h(t)^2\rangle_{main} = \frac{1}{2}\rho^3\delta_{dc}^2 \int_f df \int_{\hat{n}} d^2\hat{n} \int_{m_{1,min}}^{m_{1,max}} \int_{m_{2,min}}^{m_{2,max}} \int_{m_{3,min}}^{m_{3,max}} \Big(\frac{\Pi(m_1)}{m_1}dm_1 \frac{\Pi(m_2)}{m_2}dm_2 \frac{\Pi(m_3)}{m_3}dm_3\Big)\Big(4\pi \int_{r_2} r_2^2 e^{-N(r_2)}dr_2\Big) 4\pi \int_{r_1} r_1^2 dr_1 \Big(2\frac{F}{4}\frac{2\,G^{\frac{5}{3}}}{D(m_1,m_2,r_1)\,c^4}\Big)^2 \Big(2\frac{m_1 m_2}{(m_1+m_2)^{1/3}}\omega^{\frac{2}{3}}\Big)^2\,, \tag{2.28}$$

and similarly, for the three correction terms.
The contribution of the main term to the spectral density, after carrying out the integrations over

r_2 and r_1, (neglecting terms containing a_{eq}, as the order of $a_{eq} << 1$) is given by,

$$S_h(f)_{main} = \frac{1}{2}\rho^3\delta_{dc}^2(4\pi)\Big(\frac{c}{H_0}\Big)^{-2}\int_{m_{1,min}}^{m_{1,max}}\int_{m_{2,min}}^{m_{2,max}}\int_{m_{3,min}}^{m_{3,max}} \Big(\frac{\Pi(m_1)}{m_1}dm_1\frac{\Pi(m_2)}{m_2}dm_2\frac{\Pi(m_3)}{m_3}dm_3\Big)\Big(2\,\frac{G^{\frac{5}{3}}}{c^4}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\omega^{\frac{2}{3}}\Big)^2 \frac{4\pi}{\mathscr{D}^2\mathscr{N}(m)}\Big[\frac{exp(-\frac{4}{3}\pi\,\tilde{r}^3\mathscr{N}(m))}{\mathscr{N}(m)}\Big\{-\frac{2\alpha}{4\pi}\,a_{eq}^{1/2}-\frac{\alpha^2}{4\pi}\Big\} +\frac{2\sqrt{3}\alpha}{16\pi\mathscr{N}(m)^{3/2}}\frac{a_{eq}^{1/2}}{\tilde{r}^{3/2}}(Erf[2\sqrt{\frac{\pi}{3}}\tilde{r}^{3/2}\sqrt{\mathscr{N}(m)}]-Erf[0])]\frac{1}{4\pi\mathscr{N}(m)}\Big], \tag{2.29}$$

where the $Erf[]$ denotes the error function. Similarly, the three corrections to the spectral density are obtained from the correction terms in $h(t)$, *viz.*, for $2\Big(2\Big\{\frac{d}{dt}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\Big\}\omega^{-\frac{1}{3}}\Big)^2$, $\Big(\Big\{\frac{d^2}{dt^2}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\Big\}\omega^{-\frac{4}{3}}\Big)^2$ and $4\Big\{\frac{d^2}{dt^2}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\Big\}\omega^{-\frac{4}{3}}\Big(-2\frac{m_1m_2}{(m_1+m_2)^{1/3}}\omega^{\frac{2}{3}}\Big)$, the contributions to $S_h(f)$ are respectively :

$$S_h(f)_{1st} = \frac{1}{2}\rho^3\delta_{dc}^2(4\pi)\Big(\frac{c}{H_0}\Big)^{-2}\int_{m_{1,min}}^{m_{1,max}}\int_{m_{2,min}}^{m_{2,max}}\int_{m_{3,min}}^{m_{3,max}} \Big(\frac{\Pi(m_1)}{m_1}dm_1\frac{\Pi(m_2)}{m_2}dm_2\frac{\Pi(m_3)}{m_3}dm_3\Big)\Big(2\,\frac{G^{\frac{5}{3}}}{c^4}\Big\{\frac{d}{dt}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\Big\}\omega^{-\frac{1}{3}}\Big)^2 \frac{4\pi}{\mathscr{D}^2\mathscr{N}(m)}\Big[\frac{exp(-\frac{4}{3}\pi\,\tilde{r}^3\mathscr{N}(m))}{\mathscr{N}(m)}\Big\{-\frac{2\alpha}{4\pi}\,a_{eq}^{1/2}-\frac{\alpha^2}{4\pi}\Big\} +\frac{2\sqrt{3}\alpha}{16\pi\mathscr{N}(m)^{3/2}}\frac{a_{eq}^{1/2}}{\tilde{r}^{3/2}}(Erf[2\sqrt{\frac{\pi}{3}}\tilde{r}^{3/2}\sqrt{\mathscr{N}(m)}]-Erf[0])]+\frac{1}{4\pi\mathscr{N}(m)}\Big], \tag{2.30}$$

$$S_h(f)_{2nd} = \frac{1}{2}\rho^3\delta_{dc}^2(4\pi)\Big(\frac{c}{H_0}\Big)^{-2}\int_{m_{1,min}}^{m_{1,max}}\int_{m_{2,min}}^{m_{2,max}}\int_{m_{3,min}}^{m_{3,max}} \Big(\frac{\Pi(m_1)}{m_1}dm_1\frac{\Pi(m_2)}{m_2}dm_2\frac{\Pi(m_3)}{m_3}dm_3\Big)\frac{1}{2}\Big(\frac{G^{\frac{5}{3}}}{c^4}\Big\{\frac{d^2}{dt^2}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\Big\}\omega^{-\frac{4}{3}}\Big)^2 \frac{4\pi}{\mathscr{D}^2\mathscr{N}(m)}\Big[\frac{exp(-\frac{4}{3}\pi\,\tilde{r}^3\mathscr{N}(m))}{\mathscr{N}(m)}\Big\{-\frac{2\alpha}{4\pi}\,a_{eq}^{1/2}-\frac{\alpha^2}{4\pi}\Big\} +\frac{2\sqrt{3}\alpha}{16\pi\mathscr{N}(m)^{3/2}}\frac{a_{eq}^{1/2}}{\tilde{r}^{3/2}}(Erf[2\sqrt{\frac{\pi}{3}}\tilde{r}^{3/2}\sqrt{\mathscr{N}(m)}]-Erf[0])]+\frac{1}{4\pi\mathscr{N}(m)}\Big], \tag{2.31}$$

$$S_h(f)_{cross} = \frac{1}{2}\rho^3\delta_{dc}^2(4\pi)\Big(\frac{c}{H_0}\Big)^{-2}\int_{m_{1,min}}^{m_{1,max}}\int_{m_{2,min}}^{m_{2,max}}\int_{m_{3,min}}^{m_{3,max}} \Big(\frac{\Pi(m_1)}{m_1}dm_1\frac{\Pi(m_2)}{m_2}dm_2\frac{\Pi(m_3)}{m_3}dm_3\Big)4\Big(\frac{G^{\frac{5}{3}}}{c^4}\Big)^2\Big\{\frac{d^2}{dt^2}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\Big\}\Big(-\frac{m_1m_2}{(m_1+m_2)^{1/3}}\Big)\omega^{-\frac{2}{3}} \frac{4\pi}{\mathscr{D}^2\mathscr{N}(m)}\Big[\frac{exp(-\frac{4}{3}\pi\,\tilde{r}^3\mathscr{N}(m))}{\mathscr{N}(m)}\Big\{-\frac{2\alpha}{4\pi}\,a_{eq}^{1/2}-\frac{\alpha^2}{4\pi}\Big\} +\frac{2\sqrt{3}\alpha}{16\pi\mathscr{N}(m)^{3/2}}\frac{a_{eq}^{1/2}}{\tilde{r}^{3/2}}(Erf[2\sqrt{\frac{\pi}{3}}\tilde{r}^{3/2}\sqrt{\mathscr{N}(m)}]-Erf[0])]+\frac{1}{4\pi\mathscr{N}(m)}\Big]. \tag{2.32}$$

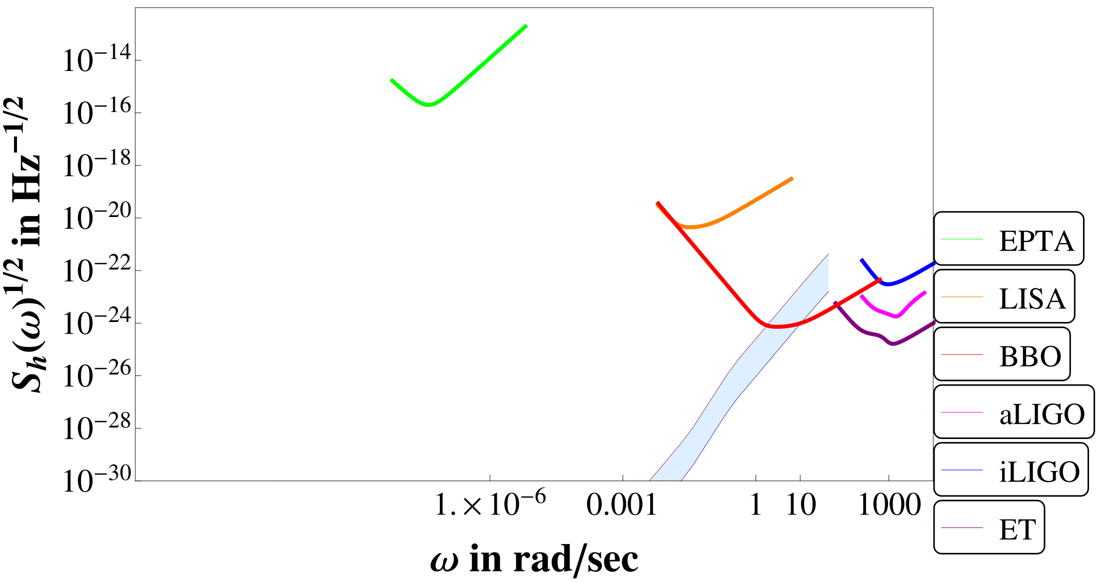

Figure 2.5: Plot of the strain sensitivity $S_h(\omega)^{1/2}$ in $Hz^{-1/2}$ of the main term vs the angular frequency (observed) ω : the band ranges for amplitude of mass-variance of primordial fluctuation ϵ from 0.1 to 0.4, for the time t = $10^{-24}\, s$ to 1 s after the big-bang.

The detectability graphs are obtained by plotting the strain sensitivity $S_h(f)^{1/2}$ (in $Hz^{-1/2}$) versus observed frequency f_o (which is $(1+z)^{-1} f_s$) imposing the noise-sensitivity lines of present and future gravitational wave detectors. It is important to note that the mass-density of PBHs in the early Universe very sensitively depends on the quantity ϵ, as given by Eq.(2.4). We plot the strain sensitivities ($S_h^{1/2}$) for certain ranges of ϵ w.r.t. the observed angular frequency in the Figs. 2.5, 2.6, 2.7, and 2.8 for the four terms, i.e., the main and three correction terms respectively, with the noise sensitivity lines for present and future gravitational wave detectors. The numerical calculations are done with Mathematica (version 9). The numerical values of the quantities $\mathscr{D}$ and α are estimated by taking the values of Ω_{DE} and Ω_M as approximately 0.68 and 0.31 respectively. These plots are shown below.

We choose the range of values of the amplitude of mass-variance of primordial fluctuation ϵ, for the strain sensitivity vs observed angular-frequency band-plot to be such that the strain sensitivity (in $Hz^{-1/2}$) has the value within 10^{-12} to 10^{-30} $Hz^{-1/2}$ which is the region where the noise-curves of most of the present and future gravitational wave detectors lie. As we know, that to be detectable, the strain produced by a gravitational wave signal must be above the noise-curve of the associated detector. Only in the case of first correction term, we have extended the lower limit of the strain sensitivity in the detectability graph to 10^{-35} $Hz^{-1/2}$, because even with very high values of ϵ, we get the strain sensitivity below 10^{-29} $Hz^{-1/2}$ for the first correction term.
In the fig. 2.5, the strain sensitivity for the main term of the stochastic background has been plotted w.r.t. corresponding observed angular frequency for the range of amplitude of mass-variance of primordial fluctuation ϵ from 0.1 to 0.4. The noise-curves for different present and future gravitational wave detectors have been shown in the figures [60]. They are iLIGO (initial LIGO), aLIGO (Advanced LIGO), LISA, ET, BBO, and EPTA . We see that certain parts of the stochastic gravitational wave background due to the main term, for the specified range of ϵ, should be detectable by future gravitational wave detector BBO.

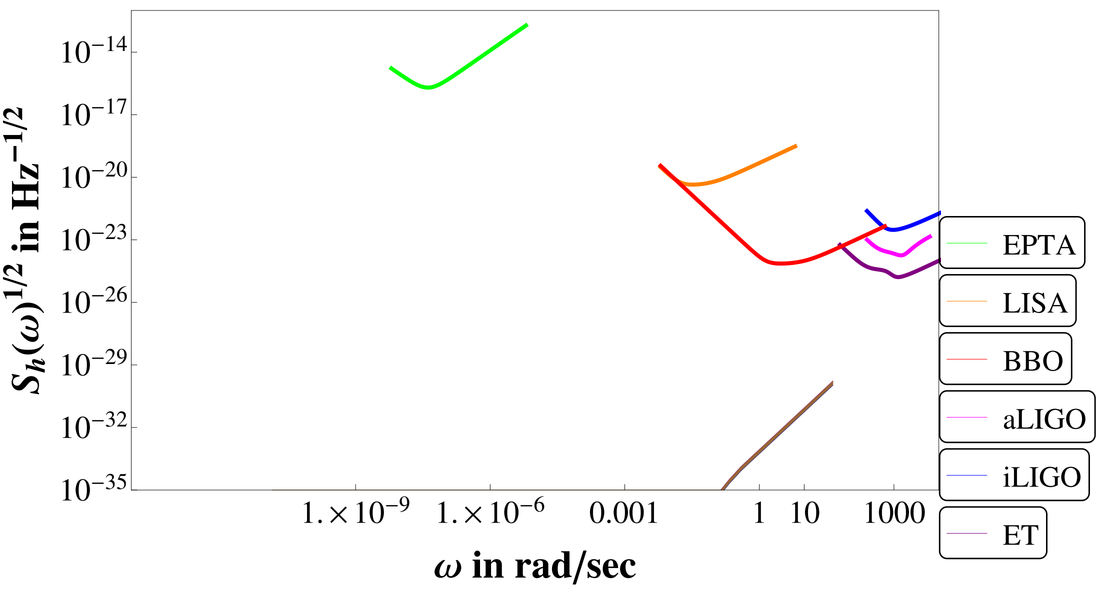

Figure 2.6: Plot of the strain sensitivity $S_h(\omega)^{1/2}$ in $Hz^{-1/2}$ of the first correction term containing the single time-derivative of the chirp mass vs the angular frequency (observed) ω : the band ranges for amplitude of mass-variance of primordial fluctuation ϵ is 0.4 to 0.8 ; for the time t $= 10^{-24}\,s$ to 1 s after the big-bang.

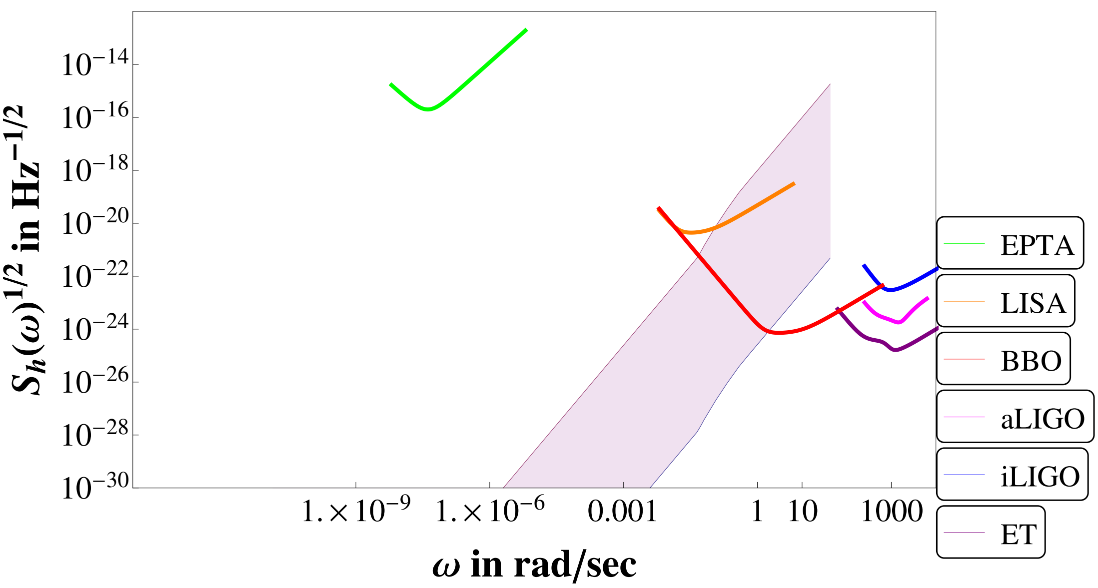

Figure 2.7: Plot of the strain sensitivity $S_h(\omega)^{1/2}$ in $Hz^{-1/2}$ of the second correction term containing double time-derivative of the chirp mass vs the angular frequency (observed) ω : the band ranges for amplitude of mass-variance of primordial fluctuation ϵ is 0.012 to 0.0125 ; for the time t $= 10^{-24}\,s$ to 1 s after the big-bang.

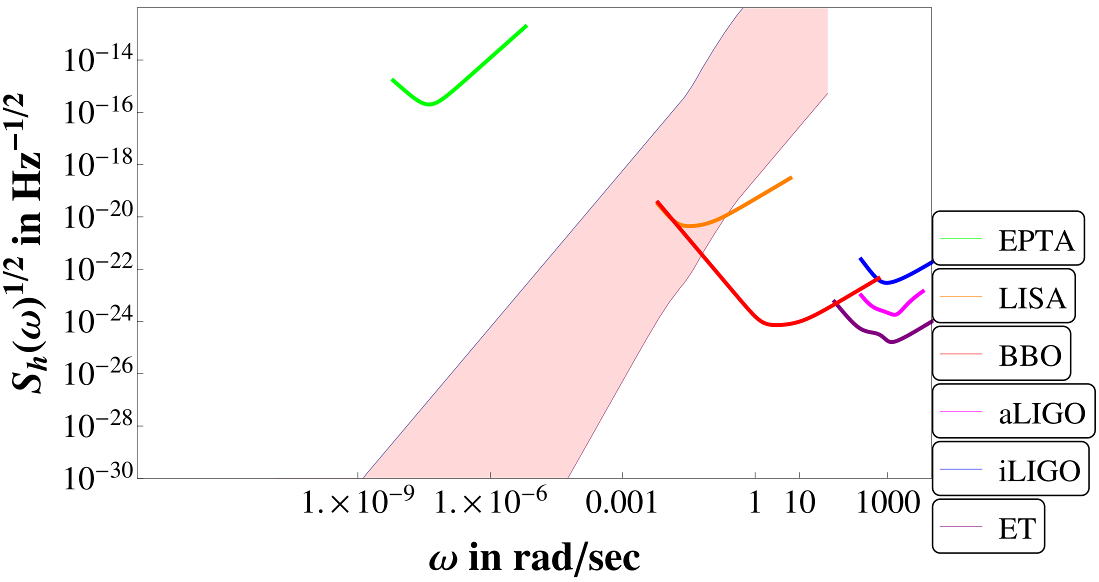

Figure 2.8: Plot of the strain sensitivity $S_h(\omega)^{1/2}$ in $Hz^{-1/2}$ of the cross-correction term due to non-vanishing product of the second correction term and main term vs the angular frequency (observed) ω : the band ranges for amplitude of mass-variance of primordial fluctuation ϵ is 0.018 to 0.02 ; for the time t = $10^{-24}\,s$ to 1 s after the big-bang.

In fig. 2.6, a similar plot of the first correction term is shown. Here, we have shown the band of strain sensitivity vs observed angular-frequency for the range of amplitude of mass-variance of primordial fluctuation ϵ from 0.4 to 0.8. It can be clearly seen that even for this range of ϵ with such high values, no region of the stochastic gravitational wave background due to the first correction term is detectable by any present or planned future gravitational wave detector.
In the fig. 2.7 and fig. 2.8, similar plots of the second correction term and the cross-correction term have been shown. In case of the figure 2.7, the band of strain sensitivity vs observed angular-frequency has been shown for the range of ϵ from 0.012 to 0.0125 and in case of figure 2.8, the range of ϵ is from 0.018 to 0.02. We see that in these cases certain portions of the stochastic gravitational wave background are detectable by LISA and BBO.
Note that the range of values of ϵ, chosen for the main term in figure 2.5, is 0.1 to 0.4, and for the first correction term in figure 2.6, is 0.4 to 0.8, which are an order larger than those for the second correction term and cross correction terms, in figures 2.7 and 2.8 respectively (where the ranges are 0.012 to 0.0125 and 0.018 to 0.02 respectively). Yet, we get greater strain for the second correction term and cross correction term than the main term and first correction term. This clearly establishes the dominance of the second and cross correction terms over the main term.

2.5 Conclusion and Discussion

In this chapter we have investigated the stochastic gravitational wave background produced by binaries of primordial black holes during their early inspiral stage while accreting high density radiation surrounding them in the early universe. It has been shown that the gravitational wave amplitude has correction terms because of the rapid rate of increase in masses of the primordial black holes. These correction terms arise due to non-vanishing first and second time derivatives of the masses and their contribution to the overall double time derivative of the quadrupole-moment tensor. We have found that some of these correction terms are not only significant in comparison with the main term, but even dominant over the main term for certain ranges of time in the early Universe. The significance of these correction terms is not only for the gravitational wave amplitude produced from an individual PBH-binary, but persists for the overall stochastic gravitational wave background produced from them.

We have further studied the detectability of the above stochastic gravitational wave background with present and future gravitational wave detectors. We find that it is possible for such contributions to the overall stochastic gravitational wave background to be directly detected with some of the future gravitational wave detectors. Moreover, it would be relevant to study the gravitational wave spectrum emitted from merger stages of such PBH binaries, which should be in the detectability range of aLIGO. Such an occurrence would thus, open up a direct window to probe the early Universe.

The significant correction terms in the spectral density generated due to rapid increase of masses of the PBHs in the binaries are explicit functions of the density of radiation at the concerned time. Hence, through these correction terms one may be able to constrain the density of radiation at a specific era in early Universe, if the stochastic background is detected in future. Moreover, observations of the stochastic background would provide direct clues of the PBH-mass ranges, rate of formation of PBH-binaries and their merging-rates, shedding light on the long-standing question as to whether some PBHs still exist in present era of our Universe comprising a fraction of the dark matter. On the other hand, if such a background is not detected, it will help setting upper limits on the PBH density in early Universe, or more fundamentally, the amplitude of mass variance of the primordial density fluctuations in the early Universe.

Chapter 3

Perturbative correction terms to electromagnetic self-force due to metric perturbation : astrophysical and cosmological implications

3.1 Introduction

The motion of a point charge in flat space-time was one of the main topics of research in physics from as early as 1930s. Many pioneering physicists like Lorentz, Abrahams, Poincare and Dirac contributed to the development of the subject from its early onset [92]. DeWitt and Brehme generalized Dirac's result to curved spacetimes, and, for the first time, gave a precise derivation of Electromagnetic self-force [93]. Later, Hobbs applied vierbein treatment to derive their equations and found that their results must be corrected by a term involving the Ricci tensor [94]. The rigorous derivation of the electromagnetic self-force was given by Samuel E. Gralla et al, in their work in 2009 [95].

Similar counterpart of electromagnetic self-force in gravity viz. the 'Gravitational self-force' also was derived, first by Mino, Sasaki, and Tanaka [96], and then by Quinn and Wald using a different method [97].

In this work, we consider an interesting case where both the electromagnetic and gravitational self-forces are present ; and both of them produce corresponding radiation reactions in the motion. We consider the equation of motion of a charged particle in curved space-time under electromagnetic radiation reaction, with external Lorentz force too and consider a physical situation such that the particle emits gravitational radiation, which perturbs the surrounding space-time. We then follow the procedure of deriving the MiSaTaQuWa equation for this equation of motion.

Here the term 'particle' does not strictly mean that it has very tiny size like elementary particles ; the mass should be centralized enough so that the equations of motion of point-particles can be applied. In this sense, a compact object like a neutron star or a stellar mass black hole orbiting a supermassive black hole can also be treated by a point-particle equation of motion. Although there are problems with point-particle notion when terms with second order metric perturbations are considered in calculation, we will not be facing it as we are considering linear metric perturbations only.

In this context the work by Peter Zimmerman and Eric Poisson [98] is very important, as it is probably the first work treating the case where electromagnetic and gravitational self-forces act together, while deriving their interaction terms. The authors in this work have considered not only the electrovac space-time, but also a more general case of scalarvac space-time, where the metric of the background space-time is a solution of the Einstein's equation in the presence a scalar field. We show that in comparison to the case where only gravitational self-force is present, the presence of electromagnetic self-force with it, adds not only the interaction terms but also several extra perturbative terms in the equation of motion. If we see the order only in terms of the charge and mass of the charged particle, then the order of these terms are proportional to q^2m for force and to q^2 for acceleration ; q being the electric charge of the particle and m being its mass. We shall discuss the issue of orders of terms in detail, in section 3.3 in this work. We interpret that these additional terms are just the perturbations to the electromagnetic self-force, due to the gravitational radiation or metric perturbations emitted by the system. We have considered these perturbative terms to the first order of the metric perturbations.

We investigate some astrophysical systems and cosmological cases, where these additional perturbative terms produced from the electromagnetic self-force, are significant in comparison with the gravitational self-force. In this case it is to be noted that although the gravitational self-force and these perturbative terms have a distinct difference to the sense that the first one is a purely gravitational aspect, while the origin of the perturbative terms are electromagnetic ; yet, they have the similarity that they contain the metric perturbations (to the linear order in this case) as they are generated due to metric perturbations. The motive of this comparison is that this will help to identify the cases where the perturbative terms generated from electromagnetic self-force would have significance or would dominate over the gravitational self-force and vice-versa. The equations of motion in the corresponding cases can be simplified or approximated accordingly.

We find that although in a few cases it is possible that these perturbative terms can be significant in comparison with the gravitational self-force, the special interest comes out to be in the cases of charged particle's motions around primordial black holes within certain mass range, which were produced in early Universe by direct gravitational collapse of sufficiently deep density perturbations.

We have discussed the orthogonality, with the four-velocity of the particle, of different terms present in the radiation reaction, in Appendix-1. There are other two Appendices. In Appendix-2, we have explained why we have neglected the perturbations originated from the term containing Ricci-tensors and in Appendix-3 we have discussed certain issues related to the comparison of different parts within the perturbative correction terms, with the gravitational self-force term.

We have not discussed the additional perturbative terms generated from the electromagnetic 'Tail term' due to the metric perturbations, in this chapter, which will be left for a separate future work. In this chapter, we have preliminarily written the electromagnetic self-forces in Gaussian units with c (speed of light in vacuum) = 1 ; but while estimating some numerical quantities related to it, we have converted this to S.I. units system.

3.2 Overcoming singularity of the retarded metric perturbation on the world line of the particle and the gauge fixing :

One main issue of our work described in this chapter is that the physical or retarded metric perturbation $h^{ret}_{\mu\nu}$ emitted from the charged particle is singular at the particle, or in other words, it is singular on the world-line of the particle. As our work deals with the world-line of the particle, we must address this singularity of the metric perturbation emitted form the particle.
For overcoming the problem of singularity of the retarded metric perturbation on the world-line of the particle, we follow the *Detweiler-Whiting formalism* [99], in which Detweiler and Whiting proposed a reformulation of the perturbed motion, where instead of breaking the overall retarded perturbation into 'Direct' and 'Tail' parts, they decomposed it into 'Singular' and 'Regular' parts :

$$h^{ret}_{\mu\nu} = h^{S}_{\mu\nu} + h^{R}_{\mu\nu}\,, \tag{3.1}$$

where the singular part $h^{S}_{\mu\nu}$ is responsible for the singular behaviour of the retarded perturbation on the world-line, while it does not generate any self-force or does not affect the motion of the particle. On the otherhand, the regular part $h^{R}_{\mu\nu}$ is a smooth solution of the perturbation equations and this exerts the identical self-force, as generated by the overall retarded perturbation. The *Detweiler-Whiting formalism* indicates the fact that the particle effectively moves along a geodesic[1] of a smooth perturbed space-time with the metric $g'_{\mu\nu} = g_{\mu\nu} + h^{R}_{\mu\nu}$[99, 100, 101], where $g_{\mu\nu}$ is the unperturbed metric.
On the world-line of the particle, the regular part of the perturbation satisfies [101]

$$\nabla_\lambda h^{R}_{\mu\nu} = -4m(u_{(\mu}R_{\nu)\rho\lambda\eta} + R_{\mu\rho\nu\eta}u_\lambda)u^\rho u^\eta + \bar{h}^{Tail}_{\mu\nu;\lambda}\,, \tag{3.2}$$

where the $\bar{h}^{Tail}_{\mu\nu;\lambda}$ is given by (in trace-reversed form) :

$$\bar{h}^{Tail}_{\mu\nu;\lambda} = 4m\int_{-\infty}^{\tau^-} \nabla_\lambda\Big(G_{+\mu\nu\mu''\nu''} - \frac{1}{2}g_{\mu\nu}G^{\rho}_{+\rho\mu''\nu''}\Big)(z(\tau), z(\tau''))u''^{\mu}u''^{\nu}d\tau''\,. \tag{3.3}$$

Therefore, we shall only work with the regular metric perturbation $h^{R}_{\mu\nu}$, thereby eliminating the singular behaviour of the retarded metric perturbation, emitted from the particle, on its world-line and all the consequent general relativistic perturbation quantities will be in terms of $h^{R}_{\mu\nu}$.
In this work, it is pertinent to be mentioned that, although in case of electromagnetic self-force, the point particle approximation is valid, but in case of general relativity, the point particle concept fails at non-linear orders. But, as the point particle concept has no problem with the linear orders of metric perturbation, which we are considering here, we can stick to this concept. However, it is worth noting that if any calculation is necessary in the non-linear orders of metric perturbation, where instead of point-particle notion the object of interest is a compact one, one has to employ a different method, known as 'Puncture method' [102, 103, 104]. In this method the retarded metric perturbation is divided into two parts known as 'Puncture part' $h^{\mathcal{P}}_{\mu\nu}$ and 'Residual part' $h^{\mathcal{R}}_{\mu\nu}$. For detailed discussion on this method, previous references [102, 103, 104] and the review article by L. Barack and A. Pound [105] can be consulted. We do not go into detail about this 'Puncture method' here, as it is not required for our analysis done with first-order metric perturbations.

[1] it is to be noted that we can use the term geodesic only in case of absence of external forces ;

Another issue in our work is that as the gravitational self-force and the metric perturbation are both gauge-dependent quantities, we have to fix the gauge to describe this physical effect meaningfully. We here choose it to be the *Lorenz-gauge*, in which many preliminary and foundational results in gravitational self-force had been obtained. In terms of the trace-reversed form of the metric perturbation :

$$\bar{h}_{\mu\nu} = h^{ret}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h^{ret} \tag{3.4}$$

($h^{ret} = g^{\alpha\beta} h^{ret}_{\alpha\beta}$), the Lorenz-gauge condition is given by :

$$\nabla^{\mu} \bar{h}_{\mu\nu} = 0\,. \tag{3.5}$$

Here, a confusion may arise that we will be working with the regular part of the perturbation $h^R_{\mu\nu}$, while the gauge-condition is in terms of the trace-reversed version of the overall perturbation $h^{ret}_{\mu\nu}$. But, there is no problem with this issue, as what happens is actually that the singular part $h^S_{\mu\nu}$ remains invariant under a smooth gauge-transformation, while only the regular part $h^R_{\mu\nu}$ changes. Furthermore, in our case, as there is electromagnetic radiation emitted from the charged particle in the space-time surrounding it and in the most general case there is also the external electromagnetic field, hence the metric actually satisfies the Einstein-Maxwell equation, not the vacuum Einstein equation. If we consider an EMRI, where the smaller component is charged, then the metric outside the larger massive body in the EMRI satisfies the Einstein's equation :

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}\,, \tag{3.6}$$

where the energy-momentum tensor $T_{\mu\nu}$ of the source outside the larger massive body contains two components : the energy-momentum tensor of the smaller charged massive body itself and the energy momentum tensor of the electromagnetic radiation emitted from the smaller charged massive body (if there be any external electromagnetic field then its energy-momentum tensor is also to be added). With the point-particle approximation, the energy momentum tensor of the smaller massive body can be represented by a Dirac-Delta function.[2] Hence, outside both the larger and smaller massive bodies, the only energy-momentum is of the electromagnetic radiation, or in other words, the metric there satisfies the Einstein-Maxwell equation :

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{\mu_o} (F_{\mu}{}^{\eta} F_{\nu\eta} - \frac{1}{4} g_{\mu\nu} F^2)\,, \tag{3.7}$$

where μ_0 is the permeability in vacuum and $F_{\mu\nu}$ is the electromagnetic field-strength tensor. So, the metric perturbations outside the larger massive body would satisfy the first order perturbation-equation of the background equation 3.7 .

3.3 Order of the perturbative terms :

Samuel E. Gralla et al gave a rigorous derivation of the electromagnetic self-force in their work in 2009 [95]. In their work, they followed an approach called "asymptotic self-similar manner". In this approach, about the world-line of the charged body as $\lambda \to 0$ (λ is a small parameter

[2] it is to be noted that this point-particle approximation of the smaller massive body and hence the corresponding Dirac-Delta function representation of its energy-momentum tensor would not be valid when second order metric perturbations would be considered, as we have already stated.

measuring the size of the charge and mass, not a parameter along the worldline [95]), the charge q and mass m of the charged body (of whose equation of motion is to be studied) tends to zero, but the charge-to-mass ratio q/m tends to a well-defined limit. This approach is considered mainly to tackle the difficulty with the point particle description of the charged body, when the limit is taken to zero-size in a straightforward way and also to avoid the problems associated with the body's finite-size consideration. As a result of following this asymptotic self-similar approach, in most of the cases the scaling of q and m are treated on equal footing i.e. it is presumed that $q \sim m$, which may not be the case in some astrophysical and cosmological scenarios.

Furthermore, when the order of a certain term in the equation of motion of the charged object is spoken about, it is often estimated and compared with the other terms only in terms of q and m. But in various cases, any other physical quantity present in that certain term, e.g. the four velocity, four acceleration and rate of change of four acceleration etc., may have such a huge order that they must have to be considered. Otherwise the estimation or comparison of the orders of these terms would turn out to be just incorrect.

In this chapter, we discuss some astrophysical and cosmological cases where the orders of these quantities like four velocity, four acceleration or rate of change of four acceleration are so huge that the overall orders of the concerned terms can not be judged only in terms of q and m. In fact, in this chapter we study all the perturbative terms which are of linear order in metric perturbation $h^R_{\alpha\beta}$, instead of designating the order of perturbation both in terms of the electromagnetic perturbation and gravitational or metric perturbation.

Generally the electromagnetic self-force is seen as a perturbation over the external Lorentz-force and hence it may seem that the electromagnetic self-force can not be compared with the gravitational force viz. the main Newtonian-part of the gravitational force. But, quite recently A. Tursunov et al has shown [106] that for a charged particle with charge q and with relativistic speed, the electromagnetic self-force, which is of the order of $\sim q^4 B^2/m^2$, can have same order of magnitude as that of Newtonian gravitational force (of the order $\sim GMm/r^2$), when the charged particle is moving around a supermassive black hole of mass $\sim 10^9 M_\odot$ ($M_\odot$ is the usual symbol of Solar-mass), in presence of a magnetic field of $B \sim 10^4$ G. [3] So, as it is possible in a practical case that the electromagnetic self-force can be of same order of magnitude with the Newtonian gravitational force, it may also be possible for the perturbations in the electromagnetic self-force caused by metric perturbations to have similar orders of magnitude as that of the gravitational self-force.

At last it is of utter importance to remember the fact that we have considered here metric perturbations originated due to the motion of the charged body itself. If there is external metric perturbation, when we speak about the orders in terms of charge and mass only, then the perturbative terms of the electromagnetic self-force originated from that external metric fluctuations would not have the order $q^2 m$, instead they would have the order q^2, same as that of the electromagnetic self-force (As the $h_{\alpha\beta}$ then would not be of the order of m). So, in that case these terms would have similar order with the gravitational self-force, which is of the order of m^2. Although in that case, for determination of the correct order would require the knowledge of the mass of that external source of the gravitational radiation. In the case of external gravitational wave, we shall not need the *Detweiler-Whiting reformulation* to break the metric perturbation into singular and regular parts, as the external metric perturbation will not be singular on the world-line of

[3] According to the works [107] and [108] The characteristic values of magnetic field near supermassive black holes of mass $\sim 10^9 M_\odot$ is B $\sim 10^4$ G .

the particle. However, in the work described in this chapter, we shall stick to the case where the gravitational radiation is generated from the charged particle itself.

3.4 The equation of motion of a charged particle in curved space-time under electromagnetic radiation reaction and applying metric perturbation to it :

Here we consider the explicit form of the equation of motion of a charged particle or compact object in curved space-time under the reaction of the electromagnetic radiation emitted by itself[101, 106] and also consider the reaction of gravitational radiation, generated due to the motion of the particle around a comparatively very bigger massive compact object, preferably a black hole. The gravitational radiation emitted from the system creates a perturbation of space-time and that would have an effect on the motion of the particle, which is the reason behind the gravitational radiation reaction. Let τ' be the proper time associated with this perturbed metric of the particle, and τ the proper time for the unperturbed metric, without the reaction of gravitational radiation. We denote the unperturbed and perturbed metric as $g_{\mu\nu}$ and $g'_{\mu\nu}$ respectively, and physical or 'retarded' part of the metric-perturbation (without the 'advanced' part of it), i.e. here the gravitational radiation emitted from the system, as $h^{ret}_{\mu\nu}$. To tackle the singularity of the metric perturbation on the world-line of the particle, we take the perturbed metric on the world-line as the effective metric : $g'_{\mu\nu} = g_{\mu\nu} + h^R_{\mu\nu}$, as we have already explained in the previous section 2. The equation of motion of the charged particle in the perturbed metric is given by :

$$\frac{Du'^{\mu}}{d\tau'} = \frac{q}{m}F'^{\mu}_{\nu}u'^{\nu} + \frac{2q^2}{3m}\Big(\frac{D^2u'^{\mu}}{d\tau'^2} + u'^{\mu}u'_{\nu}\frac{D^2u'^{\nu}}{d\tau'^2}\Big)+ \frac{q^2}{3m}(R'^{\mu}_{\lambda}u'^{\lambda} + R'^{\nu}_{\lambda}u'^{\lambda}u'^{\mu}u'_{\nu}) + \frac{2q^2}{m}f'^{\mu\nu}_{Tail}u'_{\nu}. \tag{3.8}$$

Here, $\frac{Du'^{\mu}}{d\tau'}$ is the covariant derivative of the particle's 4-velocity with respect to τ', given by

$$\frac{Du'^{\mu}}{d\tau'} = \frac{D}{d\tau'}\frac{dx^{\mu}}{d\tau'} = \frac{d^2x^{\mu}}{d\tau'^2} + \Gamma'^{\mu}_{\nu\rho}\frac{dx^{\nu}}{d\tau'}\frac{dx^{\rho}}{d\tau'}. \tag{3.9}$$

On the right-hand side of Eqn.(3.8), the first term is the Lorentz force acting on the particle, the second term is the electromagnetic radiation reaction in curved space time, the third term is due to the interaction of the particle with the surrounding matter (if there be any) and the fourth term is the 'tail term' of the electromagnetic radiation in curved space-time. The quantity $\frac{D^2u^{\mu}}{d\tau^2}$ can be expanded as [106]:

$$\frac{D^2u^{\mu}}{d\tau^2} = \frac{d^2u^{\mu}}{d\tau^2} + \partial_{\gamma}\Gamma^{\mu}_{\alpha\beta}u^{\alpha}u^{\beta}u^{\gamma} + 3\Gamma^{\mu}_{\alpha\beta}u^{\alpha}\frac{du^{\beta}}{d\tau} + \Gamma^{\mu}_{\alpha\beta}\Gamma^{\beta}_{\rho\sigma}u^{\rho}u^{\sigma}u^{\alpha}. \tag{3.10}$$

Now, we designate the contribution of the metric perturbation by an additive vector a^{μ} [4] in the equation of motion of the particle in unperturbed metric, in the following way (for the method

[4] It is to be noted very carefully that it can not be called solely a gravitational radiation reaction term. The reason for this will be clear at last, when we shall get its expression.

used here, Ref.[100] may be consulted):

$$\frac{Du^\mu}{d\tau} = \frac{q}{m}F^\mu_{\ \nu}u^\nu + \frac{2q^2}{3m}\Big(\frac{D^2u^\mu}{d\tau^2} + u^\mu u_\nu\frac{D^2u^\nu}{d\tau^2}\Big) + \frac{q^2}{3m}(R^\mu_\lambda u^\lambda + R^\nu_\lambda u^\lambda u^\mu u_\nu) + \frac{2q^2}{m}f^{\mu\nu}_{Tail}u_\nu + a^\mu. \tag{3.11}$$

Further substituting the expression of the first and second order covariant derivatives of the four-velocity of the particle, into the above Eqn.(3.10), we obtain

$$\begin{aligned}\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = \frac{q}{m}F^\mu_{\ \nu}u^\nu + \frac{2q^2}{3m}\Big(g^\mu_\eta + u^\mu u_\eta\Big)\Big(\frac{d^2u^\eta}{d\tau^2} + \partial_\gamma\Gamma^\eta_{\alpha\beta}u^\alpha u^\beta u^\gamma + \\ 3\Gamma^\eta_{\alpha\beta}u^\alpha\frac{du^\beta}{d\tau} + \Gamma^\eta_{\alpha\beta}\Gamma^\beta_{\rho\sigma}u^\rho u^\sigma u^\alpha\Big) + \frac{q^2}{3m}(R^\mu_\lambda u^\lambda + R^\nu_\lambda u^\lambda u^\mu u_\nu) + \frac{2q^2}{m}f^{\mu\nu}_{Tail}u_\nu + a^\mu.\end{aligned} \tag{3.12}$$

Next, we substitute the operators $\frac{d}{d\tau}$, $\frac{d^2}{d\tau^2}$ and $\frac{d^3}{d\tau^3}$ in the above Eqn.(3.12) with the similar ones with respect to τ' ; and obtain :

$$\begin{aligned}\frac{d^2\tau'}{d\tau^2}\frac{dx^\mu}{d\tau'} + \Big(\frac{d\tau'}{d\tau}\Big)^2\frac{d^2x^\mu}{d\tau'^2} + \Gamma^\mu_{\nu\rho}\Big(\frac{d\tau'}{d\tau}\Big)^2\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = \frac{q}{m}F^\mu_{\ \nu}\frac{d\tau'}{d\tau}\frac{dx^\nu}{d\tau'} + \frac{2q^2}{3m}\Big(g^\mu_\eta + \Big(\frac{d\tau'}{d\tau}\Big)^2\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\Big) \\ \Big[\Big\{\Big(\frac{d^3\tau'}{d\tau^3}\frac{d\tau'}{d\tau}\Big)\frac{dx^\eta}{d\tau'} + 3\Big(\frac{d\tau'}{d\tau}\frac{d^2\tau'}{d\tau^2}\Big)\frac{d^2x^\eta}{d\tau'^2} + \Big(\frac{d\tau'}{d\tau}\Big)^3\frac{d^3x^\eta}{d\tau'^3}\Big\} + \Big(\frac{d\tau'}{d\tau}\Big)^3\partial_\gamma\Gamma^\eta_{\alpha\beta}\frac{dx^\alpha}{d\tau'}\frac{dx^\beta}{d\tau'}\frac{dx^\gamma}{d\tau'} \\ +3\Gamma^\eta_{\alpha\beta}\frac{d\tau'}{d\tau}\frac{dx^\alpha}{d\tau'}\Big\{\frac{d^2\tau'}{d\tau^2}\frac{dx^\beta}{d\tau'} + \Big(\frac{d\tau'}{d\tau}\Big)^2\frac{d^2x^\beta}{d\tau'^2}\Big\} + \Gamma^\eta_{\alpha\beta}\Gamma^\beta_{\rho\sigma}\Big(\frac{d\tau'}{d\tau}\Big)^2\frac{dx^\alpha}{d\tau'}\frac{dx^\rho}{d\tau'}\frac{dx^\sigma}{d\tau'}\Big] \\ +\frac{q^2}{3m}(R^\mu_\lambda u^\lambda + R^\nu_\lambda u^\lambda u^\mu u_\nu) + \frac{2q^2}{m}f^{\mu\nu}_{Tail}\frac{d\tau'}{d\tau}\frac{dx_\nu}{d\tau'} + a^\mu.\end{aligned} \tag{3.13}$$

We substitute the expression of the $\frac{d^2x^\mu}{d\tau'^2}$ in the LHS of the above Eqn.(3.13) from the Eqn.(3.8) and hence obtain :

$$\begin{aligned}\frac{d^2\tau'}{d\tau^2}\frac{dx^\mu}{d\tau'} + \Big(\frac{d\tau'}{d\tau}\Big)^2\Big[-\Gamma'^\mu_{\nu\rho}\frac{dx^\nu}{d\tau'}\frac{dx^\rho}{d\tau'} + \frac{q}{m}F'^\mu_{\ \nu}u'^\nu + \frac{2q^2}{3m}\Big(\frac{D^2u'^\mu}{d\tau'^2} + u'^\mu u'_\nu\frac{D^2u'^\nu}{d\tau'^2}\Big) + \\ \frac{q^2}{3m}(R'^\mu_\lambda u'^\lambda + R'^\nu_\lambda u'^\lambda u'^\mu u'_\nu) + \frac{2q^2}{m}f'^{\mu\nu}_{Tail}u'_\nu\Big] + \Gamma^\mu_{\nu\rho}\Big(\frac{d\tau'}{d\tau}\Big)^2\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = \frac{q}{m}F^\mu_{\ \nu}\frac{d\tau'}{d\tau}\frac{dx^\nu}{d\tau'} + \\ \frac{2q^2}{3m}\Big(g^\mu_\eta + \Big(\frac{d\tau'}{d\tau}\Big)^2\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\Big)\Big[\Big\{\Big(\frac{d^3\tau'}{d\tau^3}\frac{d\tau'}{d\tau}\Big)\frac{dx^\eta}{d\tau'} + 3\Big(\frac{d\tau'}{d\tau}\frac{d^2\tau'}{d\tau^2}\Big)\frac{d^2x^\eta}{d\tau'^2} + \Big(\frac{d\tau'}{d\tau}\Big)^3\frac{d^3x^\eta}{d\tau'^3}\Big\} + \\ \Big(\frac{d\tau'}{d\tau}\Big)^3\partial_\gamma\Gamma^\eta_{\alpha\beta}\frac{dx^\alpha}{d\tau'}\frac{dx^\beta}{d\tau'}\frac{dx^\gamma}{d\tau'} + 3\Gamma^\eta_{\alpha\beta}\frac{d\tau'}{d\tau}\frac{dx^\alpha}{d\tau'}\Big\{\frac{d^2\tau'}{d\tau^2}\frac{dx^\beta}{d\tau'} + \Big(\frac{d\tau'}{d\tau}\Big)^2\frac{d^2x^\beta}{d\tau'^2}\Big\} \\ +\Gamma^\eta_{\alpha\beta}\Gamma^\beta_{\rho\sigma}\Big(\frac{d\tau'}{d\tau}\Big)^2\frac{dx^\alpha}{d\tau'}\frac{dx^\rho}{d\tau'}\frac{dx^\sigma}{d\tau'}\Big] + \frac{q^2}{3m}(R^\mu_\lambda u^\lambda + R^\nu_\lambda u^\lambda u^\mu u_\nu) + \frac{2q^2}{m}f^{\mu\nu}_{Tail}\frac{d\tau'}{d\tau}\frac{dx_\nu}{d\tau'} + a^\mu.\end{aligned} \tag{3.14}$$

Now we arrange the equation in such a way that the similar terms in the perturbed and unperturbed metric come together so that it can be identified. Doing so Eqn.(3.14) can be written in the following form :

$$\frac{d^2\tau'}{d\tau^2}\frac{dx^\mu}{d\tau'}+\left(\frac{d\tau'}{d\tau}\right)^2(-\Gamma'^\mu_{\nu\rho}+\Gamma^\mu_{\nu\rho})\frac{dx^\nu}{d\tau'}\frac{dx^\rho}{d\tau'}+\left(\frac{d\tau'}{d\tau}\right)^2\frac{q}{m}\left(F'^{\mu\nu}-\frac{d\tau}{d\tau'}F^{\mu\nu}\right)\frac{dx_\nu}{d\tau'}+$$
$$\left(\frac{d\tau'}{d\tau}\right)^2\frac{q^2}{3m}((R'^\mu_\lambda u'^\lambda+R'^\nu_\lambda u'^\lambda u'^\mu u'_\nu)-\left(\frac{d\tau'}{d\tau}\right)^{-2}(R^\mu_\lambda u^\lambda+R^\nu_\lambda u^\lambda u^\mu u_\nu))+$$
$$\frac{2q^2}{m}\left(\frac{d\tau'}{d\tau}\right)^2(f'^{\mu\nu}_{Tail}-\frac{d\tau}{d\tau'}f^{\mu\nu}_{Tail})u'_\nu+$$
$$\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{d^3x^\eta}{d\tau'^3}\left\{\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)-\frac{d\tau'}{d\tau}\left(g^\mu_\eta+\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)\right\}+$$
$$\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\alpha}{d\tau'}\frac{dx^\beta}{d\tau'}\frac{dx^\gamma}{d\tau'}\left\{\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)\partial_\gamma\Gamma'^\eta_{\alpha\beta}-\frac{d\tau'}{d\tau}\left(g^\mu_\eta+\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)\partial_\gamma\Gamma^\eta_{\alpha\beta}\right\}+$$
$$\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\alpha}{d\tau'}\frac{d^2x^\beta}{d\tau'^2}\left\{\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)3\Gamma'^\eta_{\alpha\beta}-\frac{d\tau'}{d\tau}\left(g^\mu_\eta+\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)3\Gamma^\eta_{\alpha\beta}\right\}+$$
$$\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\alpha}{d\tau'}\frac{dx^\rho}{d\tau'}\frac{dx^\sigma}{d\tau'}\left\{\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)\Gamma'^\eta_{\alpha\beta}\Gamma'^\beta_{\rho\sigma}-\frac{d\tau'}{d\tau}\left(g^\mu_\eta+\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)\Gamma^\eta_{\alpha\beta}\Gamma^\beta_{\rho\sigma}\right\}-$$
$$\frac{2q^2}{3m}\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)\left\{3\Gamma^\eta_{\alpha\beta}\frac{d\tau'}{d\tau}\frac{d^2\tau'}{d\tau^2}\left(\frac{dx^\alpha}{d\tau'}\frac{dx^\beta}{d\tau'}\right)+\frac{d^3\tau'}{d\tau^3}\frac{d\tau'}{d\tau}\frac{dx^\eta}{d\tau'}+3\frac{d\tau'}{d\tau}\frac{d^2\tau'}{d\tau^2}\frac{d^2x^\eta}{d\tau'^2}\right\}=a^\mu. \tag{3.15}$$

From now, we shall neglect the terms containing Ricci tensors, due to interaction with surrounding matter. The reason, for which we neglect these terms, is explained in detail in Appendix-2 of this chapter.
We simplify different terms as differences between quantities in the unperturbed metric i.e. with respect to proper time τ and in the perturbed metric i.e. with respect to proper time τ' as follows :

$$\frac{d^2\tau'}{d\tau^2}\frac{dx^\mu}{d\tau'}+\left(\frac{d\tau'}{d\tau}\right)^2(-\Delta\Gamma^\mu_{\nu\rho})\frac{dx^\nu}{d\tau'}\frac{dx^\rho}{d\tau'}-\frac{1}{2}qF^\mu_\nu u^\nu u^\alpha u^\beta h^R_{\alpha\beta}-q(g^{\mu\nu}+u^\mu u^\nu)h^R_{\nu\alpha}F^\alpha_\beta u^\beta$$
$$+\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{d^3x^\eta}{d\tau'^3}\left\{(1-\xi_1)g^\mu_\eta+h^{R\mu}_\eta+(1-\xi_1^3)\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right\}+\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\alpha}{d\tau'}\frac{dx^\beta}{d\tau'}\frac{dx^\gamma}{d\tau'}$$
$$\left\{(1-\xi_1)g^\mu_\eta\partial_\gamma\Gamma^\eta_{\alpha\beta}+g^\mu_\eta\partial_\gamma\Delta\Gamma^\eta_{\alpha\beta}+h^{R\mu}_\eta\partial_\gamma\Gamma^\eta_{\alpha\beta}+\left((1-\xi_1^3)\partial_\gamma\Gamma^\eta_{\alpha\beta}+\partial_\gamma\Delta\Gamma^\eta_{\alpha\beta}\right)\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right\}$$
$$+\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\alpha}{d\tau'}\frac{d^2x^\beta}{d\tau'^2}3\left\{(1-\xi_1)g^\mu_\eta\Gamma^\eta_{\alpha\beta}+h^{R\mu}_\eta\Gamma^\eta_{\alpha\beta}+g^\mu_\eta\Delta\Gamma^\eta_{\alpha\beta}+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\left((1-\xi_1^3)\Gamma^\eta_{\alpha\beta}+\Delta\Gamma^\eta_{\alpha\beta}\right)\right\}$$
$$+\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\alpha}{d\tau'}\frac{dx^\rho}{d\tau'}\frac{dx^\sigma}{d\tau'}\left\{(1-\xi_1)g^\mu_\eta\Gamma^\eta_{\alpha\beta}\Gamma^\beta_{\rho\sigma}+h^{R\mu}_\eta\Gamma^\eta_{\alpha\beta}\Gamma^\beta_{\rho\sigma}+g^\mu_\eta(\Delta\Gamma^\eta_{\alpha\beta})\Gamma^\beta_{\rho\sigma}+g^\mu_\eta\Gamma^\eta_{\alpha\beta}(\Delta\Gamma^\beta_{\rho\sigma})\right.$$
$$\left.+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\left((1-\xi_1^3)\Gamma^\eta_{\alpha\beta}\Gamma^\beta_{\rho\sigma}+\Gamma^\beta_{\rho\sigma}\Delta\Gamma^\eta_{\alpha\beta}+\Gamma^\eta_{\alpha\beta}\Delta\Gamma^\beta_{\rho\sigma}\right)\right\}+\frac{2q^2}{m}\left(\frac{d\tau'}{d\tau}\right)^2(f'^{\mu\nu}_{Tail}-\frac{d\tau}{d\tau'}f^{\mu\nu}_{Tail})\frac{dx_\nu}{d\tau'}$$
$$-\frac{2q^2}{3m}\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)\left\{3\Gamma^\eta_{\alpha\beta}\frac{d\tau'}{d\tau}\frac{d^2\tau'}{d\tau^2}\left(\frac{dx^\alpha}{d\tau'}\frac{dx^\beta}{d\tau'}\right)+\frac{d^3\tau'}{d\tau^3}\frac{d\tau'}{d\tau}\frac{dx^\eta}{d\tau'}+3\frac{d\tau'}{d\tau}\frac{d^2\tau'}{d\tau^2}\frac{d^2x^\eta}{d\tau'^2}\right\}=a^\mu, \tag{3.16}$$

where the terms $-\frac{1}{2}qF^\mu_\nu u^\nu u^\alpha u^\beta h^R_{\alpha\beta}$ and $-q(g^{\mu\nu}+u^\mu u^\nu)h^R_{\nu\alpha}F^\alpha_\beta u^\beta$ have already been derived by P. Zimmerman and E. Poisson in an earlier work [98].
For brevity, we introduced the quantity

$$\xi_1=\frac{d\tau'}{d\tau} \tag{3.17}$$

In the above equation the quantity $\Delta\Gamma^{\mu}_{\nu\rho}$ is the part of the perturbation in the Christoffel symbol tensor, caused by the regular part $h^{R}_{\mu\nu}$ of the metric perturbation and is given by :

$$\Delta\Gamma^{\mu}_{\nu\rho} = \Gamma'^{\mu}_{\nu\rho} - \Gamma^{\mu}_{\nu\rho} = \frac{1}{2}h^{R\,\mu\alpha}(\partial_{\rho}g_{\alpha\nu} + \partial_{\nu}g_{\alpha\rho} - \partial_{\alpha}g_{\nu\lambda}) + \frac{1}{2}g^{\mu\alpha}(\partial_{\rho}h^{R}_{\alpha\nu} + \partial_{\nu}h^{R}_{\alpha\rho} - \partial_{\alpha}h^{R}_{\nu\lambda}). \tag{3.18}$$

The above expression of $\Delta\Gamma^{\mu}_{\nu\rho}$ can be further simplified as :

$$\Delta\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}g^{\mu\alpha}(\nabla_{\rho}h^{R}_{\alpha\nu} + \nabla_{\nu}h^{R}_{\alpha\rho} - \nabla_{\alpha}h^{R}_{\nu\lambda}). \tag{3.19}$$

3.5 Extra terms generated due to perturbation of the electromagnetic self-force and their significance :

In this section, we investigate the significance of the additional perturbative terms generated due to metric perturbations from the electromagnetic self-force in comparison with the gravitational self-force. The gravitational radiation reaction on the motion of the particle in curved space time in the absence of electromagnetic radiation reaction is given by :

$$a^{\mu}_{1} = \frac{d^{2}\tau'}{d\tau^{2}}\frac{d\tau}{d\tau'}u^{\mu} - \Delta\Gamma^{\mu}_{\nu\rho}u^{\nu}u^{\rho} = -\xi_{1}^{2}(\delta^{\mu}_{\eta} + u^{\mu}u_{\eta})\Delta\Gamma^{\eta}_{\alpha\beta}u'^{\alpha}u'^{\beta}. \tag{3.20}$$

It is to be noted that the above simplified expression of the gravitational radiation reaction term can be obtained by applying the orthogonality property of the reaction, in the case where there is no electromagnetic radiation reaction [100].
Now, the additional perturbative terms, in linear order of the metric perturbation $h^{R}_{\alpha\beta}$, generated from the electromagnetic radiation reaction due to the metric perturbations emitted from the particle, which are absent when there is only one among electromagnetic self-force and metric-perturbations, are :

$$a^{\mu}_{int\,1} = -\frac{1}{2}\frac{q}{m}F^{\mu}_{\ \nu}u^{\nu}u^{\alpha}u^{\beta}h^{R}_{\alpha\beta}\,, \tag{3.21}$$

$$a^{\mu}_{int\,2} = -\frac{q}{m}(g^{\mu\nu} + u^{\mu}u^{\nu})h^{R}_{\nu\alpha}F^{\alpha}_{\ \beta}u^{\beta}\,, \tag{3.22}$$

$$a^{\mu}_{2} = \frac{2q^{2}}{3m}\xi_{1}^{2}h^{R\,\mu}_{\ \ \eta}\frac{d^{2}u'^{\eta}}{d\tau'^{2}}\,, \tag{3.23}$$

$$a^{\mu}_{3} = \frac{2q^{2}}{m}\xi_{1}^{2}u'^{\alpha}\frac{du'^{\beta}}{d\tau'}(h^{R\,\mu}_{\ \ \eta}\Gamma^{\eta}_{\alpha\beta} + \Delta\Gamma^{\mu}_{\alpha\beta} + \Delta\Gamma^{\eta}_{\alpha\beta}u'^{\mu}u'_{\eta})\,, \tag{3.24}$$

$$a^{\mu}_{4} = \frac{2q^{2}}{3m}\xi_{1}^{2}u'^{\alpha}u'^{\beta}u'^{\gamma}(\partial_{\gamma}\Delta\Gamma^{\mu}_{\alpha\beta} + h^{R\,\mu}_{\ \ \eta}\partial_{\gamma}\Gamma^{\eta}_{\alpha\beta} + u'^{\mu}u'_{\eta}\partial_{\gamma}\Delta\Gamma^{\eta}_{\alpha\beta})\,, \tag{3.25}$$

and

$$a^{\mu}_{5} = \frac{2q^{2}}{3m}\xi_{1}^{2}u'^{\alpha}u'^{\rho}u'^{\sigma}(h^{R\,\mu}_{\ \ \eta}\Gamma^{\eta}_{\alpha\beta}\Gamma^{\beta}_{\rho\sigma} + \Gamma^{\beta}_{\rho\sigma}\Delta\Gamma^{\mu}_{\alpha\beta} + \Gamma^{\mu}_{\alpha\beta}\Delta\Gamma^{\beta}_{\rho\sigma} + u'^{\mu}u'_{\eta}(\Gamma^{\beta}_{\rho\sigma}\Delta\Gamma^{\eta}_{\alpha\beta} + \Gamma^{\eta}_{\alpha\beta}\Delta\Gamma^{\beta}_{\rho\sigma})). \tag{3.26}$$

In this case, there may be a confusion that why we have written these correction terms in equations 3.23 to 3.26 separately, although their basic-source is same : the second term on the R.H.S. of the equation 3.8 i.e. the Abraham-Lorentz-Dirac term. The simple reason behind this is that despite having identical source, these terms originate from four different kind of terms within the Abraham-Lorentz-Dirac term. This can be clearly checked from the equation 3.9, where we have written the detailed expression of $\frac{D^2u^\mu}{d\tau^2}$, expanding the covarinat-derivatives within it.
In the next subsections we analyze the significance of the terms a_3^μ, a_4^μ and a_5^μ with respect to the gravitational radiation reaction term a_1^μ. We shall avoid discussing the significance of the term a_2^μ, as it contains time-rate of change of acceleration and hence this is very complicated to compare for practical astrophysical and cosmological phenomena.

The significance of the term a_3^μ : Let us now analyze the ratio :

$$\frac{a_3^\mu}{a_1^\mu} = \frac{\dfrac{2q^2}{m}\xi_1^2 u'^\alpha \dfrac{du'^\beta}{d\tau'}(h^{R}{}_{\eta}{}^{\mu}\Gamma^\eta_{\alpha\beta} + \Delta\Gamma^\mu_{\alpha\beta} + \Delta\Gamma^\eta_{\alpha\beta}u'^\mu u'_\eta)}{-\xi_1^2(\delta^\mu_\eta + u^\mu u_\eta)\Delta\Gamma^\eta_{\alpha\beta}u'^\alpha u'^\beta}. \tag{3.27}$$

We see that for $a_3^\mu \sim a_1^\mu$, one of the requirement is:

$$\frac{2q^2}{m}\xi_1^2 u'^\alpha \frac{du'^\beta}{d\tau'}\Delta\Gamma^\eta_{\alpha\beta}u'^\mu u'_\eta \sim \xi_1^2 u'^\alpha \Delta\Gamma^\eta_{\alpha\beta}u'^\beta u^\mu u_\eta. \tag{3.28}$$

After substituting $u^\mu u_\eta \equiv \left(\frac{d\tau'}{d\tau}\right)^2 u'^\mu u'_\eta$ and cancelling out the u'^μ from both sides of the above condition, we obtain :

$$\frac{2q^2}{m}\xi_1^2 u'^\alpha \frac{du'^\beta}{d\tau'}\Delta\Gamma^\eta_{\alpha\beta}u'_\eta \sim \xi_1^4 u'^\alpha \Delta\Gamma^\eta_{\alpha\beta}u'^\beta u'_\eta. \tag{3.29}$$

It is to be noticed that the above relation is a tensorial one where on both sides three indices α, β and η are repeated indices and they are contracted among the tensors in such a way that we can not cancel out the term $\Delta\Gamma^\eta_{\alpha\beta}u'^\alpha u'_\eta$ from both sides of Eqn.(3.29), although that is common. For that, we write the expanded expression of the quantity $\Delta\Gamma^\eta_{\alpha\beta}\dfrac{du'^\beta}{d\tau'}u'^\alpha u'_\eta$ with respect to the repeated index β :

$$\Delta\Gamma^\eta_{\alpha\beta}\frac{du'^\beta}{d\tau'}u'^\alpha u'_\eta = \Delta\Gamma^\eta_{\alpha r}\frac{du'^r}{d\tau'}u'^\alpha u'_\eta + \Delta\Gamma^\eta_{\alpha\theta}\frac{du'^\theta}{d\tau'}u'^\alpha u'_\eta +$$
$$\Delta\Gamma^\eta_{\alpha\phi}\frac{du'^\phi}{d\tau'}u'^\alpha u'_\eta + \Delta\Gamma^\eta_{\alpha t}\frac{du'^t}{d\tau'}u'^\alpha u'_\eta\ .$$

Where the indices r, θ, ϕ and t denote the four coordinates of the coordinate system. In a similar way, we expand the quantity $\Delta\Gamma^\eta_{\alpha\beta}u'^\beta u'_\eta u'^\alpha$ with respect to β, and then substituting the expanded forms of these two quantities in both sides of the relation (3.29) to obtain the condition to be satisfied in coordinate-wise manner, we get [5] :

$$\frac{2q^2}{m}\xi_1^2 u'^\alpha \frac{du'^r}{d\tau'}\Delta\Gamma^\eta_{\alpha r}u'_\eta \sim \xi_1^4 u'^\alpha \Delta\Gamma^\eta_{\alpha r}u'^r u'_\eta, \tag{3.30}$$

$$\frac{2q^2}{m}\xi_1^2 u'^\alpha \frac{du'^\theta}{d\tau'}\Delta\Gamma^\eta_{\alpha\theta}u'_\eta \sim \xi_1^4 u'^\alpha \Delta\Gamma^\eta_{\alpha\theta}u'^\theta u'_\eta, \tag{3.31}$$

$$\frac{2q^2}{m}\xi_1^2 u'^\alpha \frac{du'^\phi}{d\tau'}\Delta\Gamma^\eta_{\alpha\phi}u'_\eta \sim \xi_1^4 u'^\alpha \Delta\Gamma^\eta_{\alpha\phi}u'^\phi u'_\eta, \tag{3.32}$$

$$\frac{2q^2}{m}\xi_1^2 u'^\alpha \frac{du'^t}{d\tau'}\Delta\Gamma^\eta_{\alpha t}u'_\eta \sim \xi_1^4 u'^\alpha \Delta\Gamma^\eta_{\alpha t}u'^t u'_\eta. \tag{3.33}$$

Let us consider any one of the above set of relations, say the first one based on radial coordinate r. It gives :

$$\frac{2q^2}{m}\frac{du'^r}{d\tau'} \sim \xi_1^2 u'^r; \tag{3.34}$$

provided that the quantity $\Delta\Gamma^\eta_{\alpha r}u'_\eta u'^\alpha$ is non-zero, which should be obvious for any curved space time as any of the spatial components of the four-velocity must be non-zero for the motion of the particle or compact object, as well as any of the perturbed component of the Christoffel. Thus, the condition (3.34) in S.I. units reads:[6]

$$\frac{q^2}{2\pi\epsilon_0 c^3 m}\frac{du'^r}{d\tau'} \sim u'^r. \tag{3.35}$$

We describe the fulfillment of the condition (3.35) in two different classes of charged objects : (i) charged sub-atomic particles (e.g. we estimate it numerically for a proton) and (ii) charged neutron stars or stellar mass black holes. If we consider the orbiting particle to be a proton, the above condition yields

$$\frac{\left(\frac{du'^r}{d\tau'}\right)}{u'^r} \sim 10^{26}\, s^{-1}. \tag{3.36}$$

Therefore, for a proton orbiting around a black hole the condition for the significance of the perturbative term a_3^μ is that the acceleration of the proton has to be 10^{26} order larger than the speed [7].

It would be interesting to test the significance of the term a_3^μ in the cases of charged stars, specially charged neutron stars or white dwarfs, or charged stellar mass black holes revolving around a supermassive black hole. Although, still there is no distinct astronomical evidence for any compact object containing a significant amount of net electric charge, specially for stars and black holes, many researchers have been working on theoretical models of stars containing a significant amount of net electric charge. For instance, in the reference [109] the authors have discussed a class of static stellar equilibrium configurations of relativistic spheres made of charged perfect fluids, where they have analyzed the physical acceptability of their theoretical model for

[5] Although the set of conditions (3.30) to (3.33) together obviously satisfy the condition (3.29), it is not necessarily the only case for satisfying this condition, as the terms in the above equations are summed up in (3.29). We take the condition in coordinate-wise for brevity of our analysis.

[6] As, we have considered the metric perturbation to be sufficiently small, so that the perturbation terms are retained up to linear order only ; hence the difference between the corresponding proper times of perturbed and unperturbed metrics viz. τ' and τ should also be small enough such that $\frac{d\tau'}{d\tau} \sim 1$).

[7] Here we use proton just to get an idea on the order of acceleration required for such sub-atomic particles to satisfy the condition 3.35.

some compact star candidates like SAX J1808.4-3658, 4U 1538-52, PSR J1903+327, Vela X-1 and 4U1608-52. They have concluded that their results strongly suggest that a class of compact stellar models with charged perfect fluid matter distribution is permitted with the new solution discussed in their work.

According to the references [110, 111, 112], and [113], the global balance of forces allows a net charge as large as $10^{20}\,C$ in neutron stars, producing a very high electric field of order $\sim 10^{21}\,V/m$. Then the condition (5.27) for that would be :

$$\frac{du'^{\beta}}{d\tau'} \sim \left(\frac{M_{NS}}{M_{\odot}} \times 10^{6}\,s^{-1}\right) u'^{\beta}, \tag{3.37}$$

where M_{NS} and $M_{\odot}$ denote the mass of the neutron star and the solar mass, respectively. We know that usually neutron stars and white dwarfs have mass of the order of solar mass $M_{\odot}$ i.e. $M_{NS}/M_{\odot} \sim 1$. Here, we consider the observational work in the reference [114], which reports that the speed of a star around the supermassive black hole at the center of Milky-way reaches approximately $8 \times 10^{6}\,ms^{-1}$, and hence, in such cases, for satisfying the condition derived above, the acceleration of the star around the supermassive black hole has to be $\sim 10^{12}\,ms^{-2}$. So, achieving acceleration of this order would be quite difficult if a single neutron star or stellar mass black hole revolves around a supermassive black hole. To satisfy the condition (3.37), a different type of astrophysical configuration is required. When a stellar mass black hole binary or a neutron star binary or a neutron star-black hole binary would be revolving around a supermassive black hole, then the smaller components in binary formation within the three-body system should achieve the required acceleration to satisfy the condition (3.37). It has been shown in the work by Xian Chen et al [115], that such EMRIs are expected to be produced by tidal capture of smaller binaries by a supermassive black hole. Again, at the late inspiral stage or at the merging stage, this binary becomes sufficiently compact with respect to the supermassive black hole such that it can be treated with the point-particle equations in our work.

In support of the fact that the smaller components of these EMRIs can achieve such order of acceleration required for satisfaction of the condition (3.37), we give an example of the acceleration in the binary black hole candidate GW150914, from which first direct detection of gravitational waves by aLIGO had been done [5]. For this candidate, during 0.2 second time interval of the detectable gravitational wave signal, the relative orbiting velocity of the black holes increased from 30% to 60% of the speed of light, and hence in this case the order of acceleration was approximately $10^{8}\,m.s^{-2}$. Although the merger-stage dynamics of the black holes can be accurately determined by numerical general relativistic techniques only, yet from this estimation of acceleration, we get an intuitive idea that in case of typical binaries of stellar mass black holes or neutron stars or binaries of black hole-neutron star, the acceleration achieved in the late inspiral stage and merger stage would be of similar order or even more. So, if the smaller component of an EMRI be such a binary, where the stars or stellar mass black holes contain sufficient net electric charge, then satisfying the condition (3.37) is clearly possible.

Therefore, binary formations of neutron stars or stellar mass black holes containing a net electric charge of order $10^{20}\,C$ and inspiralling around supermassive black holes, are expected to satisfy the condition of significance (3.37). This type of extreme mass-ratio inspirals (EMRIs) are expected to be detected by the upcoming space-based gravitational wave detector LISA. Hence, if the smaller mass components of such EMRIs contain a significant amount of net charge, then neglecting the term a_3^{μ}, generated due to perturbation of electromagnetic radiation reaction by

metric fluctuations or gravitational radiation, may lead to theoretically wrong estimation of the parameters related to these sources of gravitational waves.
Next, we consider a period of early universe where primordial black holes (PBHs) are expected to be produced by direct gravitational collapse of sufficiently deep density perturbations. Furthermore, during this epoch, as the universe was full of charged particles (i.e. atom formation did not start yet), it is unlikely that the PBHs would be neutral. Therefore, in this case charged particles would inspiral around charged PBHs and ultimately fall into the PBHs, emitting gravitational and electromagnetic radiation. The interesting fact here is that many of these systems could have the size of atoms. For instance, the PBHs which had mass smaller than $\sim 10^{20}\,kg$, their Schwarzschild radius would be less than $10^{-7}\,m$, and thus, we expect that they can be treated as quantum particles. Therefore, in such systems we expect that the charged particles orbiting around charged PBHs would have huge acceleration, as required by the condition (3.36), and consequently, the perturbative term a_3^μ is expected to be significant[8].
Another astrophysical phenomenon where the condition (3.36) may be satisfied is in 'Relativistic Astrophysical Jets'. In these case, accelerated ionized matter are emitted in the form of a beam from some high-energy astrophysical sources and usually the magnitude of their acceleration is huge. If in any of such astrophysical jet, the ions are accelerated during sufficiently small time to relativistic speeds, then the condition (3.36) is expected to be satisfied in the part of the jet closest to the source, specially a supermassive black hole at the center of an active galaxy. Even if the accelerated ions of such a relativistic astrophysical jet passes through the vicinity of another black hole or compact object, we may also expect that the condition (3.36) to be satisfied.[9]

For $a_3^\mu \sim a_1^\mu$, another requirement is:

$$\frac{2q^2}{m}\xi_1^2 u'^\alpha \frac{du'^\beta}{d\tau'} h^{R}{}_{\eta}{}^{\mu}\Gamma^\eta_{\alpha\beta} \sim \xi_1^2 u'^\alpha u'^\beta \Delta\Gamma^\mu_{\alpha\beta}. \tag{3.38}$$

Analyzing the condition in coordinate-wise manner as we did before , we obtain for the radial coordinate:

$$\frac{2q^2}{m}\frac{du'^\beta}{d\tau'} h^{R}{}_{\eta}{}^{\mu}\Gamma^\eta_{r\beta} \sim u'^\beta \Delta\Gamma^\mu_{r\beta}. \tag{3.39}$$

At this point, we make use of Eqn.(3.16), and substitute just the first term in the expression of $\Delta\Gamma^\mu_{r\beta}$ into Eqn.(3.39), and we get

$$\frac{2q^2}{m}\frac{du'^\beta}{d\tau'} \sim u'^\beta. \tag{3.40}$$

It is interesting to note that finally we obtain the identical condition given in Eqn.(3.34) (or, equivalently, in Eqn.(3.35)), and hence the practical cases where this would be satisfied are also same.

[8] An example of a system where we can have a huge acceleration is the revolving of an electron around a nucleus in the Bohr-model of atom. The order of magnitude of such acceleration is $\sim 10^{22}\,ms^{-2}$. Hence, as in the early Universe the charged particles revolving around charged PBHs constituted systems of atomic-size, emitting both gravitational and electromagnetic radiation, there also the acceleration of the revolving particle would be quite similar, even expected to be larger due to the curvature of the PBH in comparison with nucleus of a typical atom.

[9] In any astrophysical scenario, if the the Plasma acceleration of ions can be achieved, then that would be an ideal case for satisfying the condition (3.36). Indeed, in plasma acceleration of ions, the magnitude of acceleration as high as $10^{22} - 10^{23}\,ms^{-2}$ can be reached [116, 117].

Next, we compare the left hand side of Eqn.(3.39) with the part of $\Delta\Gamma^{\mu}_{r\beta}$ involving derivatives of $h^R_{\mu\nu}$, and we find

$$\begin{aligned}\frac{2q^2}{m}\frac{du'^{\beta}}{d\tau'}h^{R\,\mu\kappa}(\partial_r g_{\kappa\beta}+\partial_\beta g_{\kappa r}-\partial_\kappa g_{r\beta}) \sim \\ u'^{\beta}g^{\mu\kappa}(\partial_r h^R_{\kappa\beta}+\partial_\beta h^R_{\kappa r}-\partial_\kappa h^R_{r\beta})\,.\end{aligned} \tag{3.41}$$

It is to be noted that whether the above condition would be satisfied in any case, would depend on the associated components of the regular part of gravitational radiation $h^{R\,\mu\kappa}$ and metric $g^{\mu\kappa}$ (more specifically saying it would depend on the index μ, as the rest of the indices are repeated indices). Here, we are giving a qualitative discussion on satisfying the condition, rather than a quantitative analysis. Actually it depends on the fact that how does the regular part of the gravitational radiation vary with radial distance from the source. On the other hand, if we consider spherically symmetric metrics, then the partial derivative with respect to r of $g_{\theta\theta}$ and $g_{\phi\phi}$ would give a factor of $2r$, while those of g_{tt} and g_{rr} would depend on the particular type of that metric. However, it is well known that if the radial distance r is not too small, then the amplitude of gravitational radiation (and the regular part of this gravitational radiation too) acting here as the metric perturbation, is very much less than that of the metric components. Hence, for typical astrophysical systems, the values of quantities $h^{R\,\mu\kappa}$ and $(\partial_r h^R_{\kappa\beta}+\partial_\beta h^R_{\kappa r}-\partial_\kappa h^R_{r\beta})$ should be very small. While, the values of quantities $g^{\mu\kappa}$ and $(\partial_r g_{\kappa\beta}+\partial_\beta g_{\kappa r}-\partial_\kappa g_{r\beta})$ are relatively very large than them. Hence, depending on the situation it is possible to get some cases where the condition (3.41) is satisfied.

The significance of the term a^{μ}_4 : Next, we take the ratio of the perturbative term a^{μ}_4 with the gravitational radiation reaction term a^{μ}_1:

$$\frac{a^{\mu}_4}{a^{\mu}_1} = \frac{\frac{2q^2}{3m}\xi_1^2 u'^{\alpha}u'^{\beta}u'^{\gamma}(\partial_\gamma\Delta\Gamma^{\mu}_{\alpha\beta}+h^{R\,\mu}_{\eta}\partial_\gamma\Gamma^{\eta}_{\alpha\beta}+u'^{\mu}u'_{\eta}\partial_\gamma\Delta\Gamma^{\eta}_{\alpha\beta})}{-\xi_1^2(\delta^{\mu}_{\eta}+u^{\mu}u_{\eta})\Delta\Gamma^{\eta}_{\alpha\beta}u'^{\alpha}u'^{\beta}}, \tag{3.42}$$

and one of the conditions for $a^{\mu}_4 \sim a^{\mu}_1$ reads

$$\frac{2q^2}{3m}u'^{\alpha}u'^{\beta}u'^{\gamma}\partial_\gamma\Delta\Gamma^{\mu}_{\alpha\beta} \sim \Delta\Gamma^{\mu}_{\alpha\beta}u'^{\alpha}u'^{\beta}. \tag{3.43}$$

Following the same steps as in the previous subsection, we find

$$\frac{2q^2}{3m}u'^{\gamma}\partial_\gamma g^{\mu r} \sim g^{\mu r}. \tag{3.44}$$

If we consider the metric to be Reissner-Nordstrom metric, then as the metric is spherically symmetric and time-independent, the above condition (3.44) yields :

$$\frac{2q^2}{3m}(u'^{r}\partial_r+u'^{\theta}\partial_\theta)g^{\mu r} \sim g^{\mu r}. \tag{3.45}$$

It is very interesting to find that not only for g^{rr}, but for all non-zero components of the Reissner-Nordstrom metric, the above condition gives in S.I. units:

$$\frac{q^2}{6\pi\epsilon_0 c^3 m}|u'^{r}| \sim r, \tag{3.46}$$

where r is the radial distance from the black hole. For protons, the above condition yields

$$|u'^r| \sim 10^{26} s^{-1}\, r(in\, m). \tag{3.47}$$

Therefore, we expect the term a_4^μ to be significant for ultra-relativistic motion of protons around PBHs of mass less than $10^9\, kg$, which is equivalent to Schwarzschild length scale of $10^{-18}\, m$.

The significance of the term a_5^μ : Now, we check the significance of the term a_5^μ with respect to the gravitational radiation reaction term a_1^μ. Like the previous two terms, we first observe the ratio :

$$\frac{a_5^\mu}{a_1^\mu} = \frac{\frac{2q^2}{3m}\xi_1^2 u'^\alpha u'^\rho u'^\sigma (h^{R\,\mu}_{\eta}\Gamma^\eta_{\alpha\beta}\Gamma^\beta_{\rho\sigma} + \Gamma^\beta_{\rho\sigma}\Delta\Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\alpha\beta}\Delta\Gamma^\beta_{\rho\sigma} + u'^\mu u'_\eta(\Gamma^\beta_{\rho\sigma}\Delta\Gamma^\eta_{\alpha\beta} + \Gamma^\eta_{\alpha\beta}\Delta\Gamma^\beta_{\rho\sigma}))}{-\xi_1^2(\delta^\mu_\eta + u^\mu u_\eta)\Delta\Gamma^\eta_{\alpha\beta}u'^\alpha u'^\beta}. \tag{3.48}$$

So, one of the conditions for $a_5^\mu \sim a_1^\mu$ is given by:

$$\frac{2q^2}{3m}\xi_1^2 u'^\alpha u'^\rho u'^\sigma u'^\mu u'_\eta \Gamma^\beta_{\rho\sigma}\Delta\Gamma^\eta_{\alpha\beta} \sim \xi_1^2 u^\mu u_\eta \Delta\Gamma^\eta_{\alpha\beta}u'^\alpha u'^\beta. \tag{3.49}$$

Following the similar steps we carried out previously, we explore this condition in coordinate-wise manner, for the Reissner-Nordstrom black hole carrying constant charge[10], and after some algebra we obtain

$$\frac{q^2}{3m}(u'^r)^2 g^{rr}\partial_r g_{rr} \sim \xi_1^2 u'^r. \tag{3.50}$$

Taking $\xi \approx 1$, and using $g_{rr} = (1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2})^{-1}$, the above condition becomes[11]

$$\frac{q^2}{3m}u'^r \frac{\frac{1}{r}\left(\frac{r_s}{r} - \frac{2r_Q^2}{r^2}\right)}{\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)} \sim 1. \tag{3.51}$$

Where, r_s is the Schwarzschild radius of the black hole given by $r_s = 2GM/c^2$ and r_Q is the length scale associated with the electrical charge Q of the black hole, given by $r_Q = Q^2G/4\pi\epsilon_0 c^4$. For practical cases of charged particles or compact objects moving around charged black holes, [12] r is larger than both these length scales. However, the situation can be classified into two different cases : (i) $r > r_s$, r_Q, but yet $r \sim r_s$, r_Q, and (ii) $r >> r_s, r_Q$. In the former case, the quantity $\left(\frac{r_s}{r} - \frac{2r_Q^2}{r^2}\right)/\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)$ is of order unity, whereas in the later case it would be very large. We here focus on case (i), because if the particle moves very far away from the black hole then the effect of gravitational wave on its motion would be very little. Thus, condition (3.51) reduces

[10] Although in many cosmological cases, the charged black holes would have charges varying continuously with time, as of now we consider black holes with constant charges.

[11] In this condition 3.51, on the LHS, we have considered the magnitude of the quantity only, neglecting the sign.

[12] Here we do not specify the sign of the charge of the particles. In fact, even if the particles' charge are of the same sign as that of the black hole, they may gravitationally bounded by the charged black hole if the gravitational pull due to space-time curvature overtakes the effect of electrostatic repulsion[118], [119], [120], [121].

(in S.I. units) to

$$\frac{q^2}{12\pi\epsilon_0 c^3 m}|u'^r| \sim r. \tag{3.52}$$

This condition is similar to the one in Eqn.(3.46), and, hence, the cases of satisfaction of this condition would also be identical as discussed earlier.
Another condition of significance of a_5^μ with respect to a_1^μ can be given by :

$$\frac{2q^2}{3m}\xi_1^2 u'^\alpha u'^\rho u'^\sigma \Gamma^\beta_{\rho\sigma}\Delta\Gamma^\mu_{\alpha\beta} \sim \xi_1^2 u'^\alpha u'^\beta \Delta\Gamma^\mu_{\alpha\beta}. \tag{3.53}$$

We expand the repeated index β on both sides of the above condition, and follow similar steps using the condition coordinate-wise, as described in previous cases. Then we take the condition for the radial coordinate for the repeated-index β and from that, cancelling the quantity $u'^\alpha\Delta\Gamma^\mu_{\alpha r}$ from both sides, we get

$$\frac{2q^2}{3m}u'^\rho u'^\sigma \Gamma^r_{\rho\sigma} \sim u'^r. \tag{3.54}$$

If we expand the sum denoted by contracted repeated index ρ, in the LHS of above condition (3.54), then we obtain :

$$u'^\sigma \Gamma^r_{r\sigma} \sim \frac{3m}{2q^2} \tag{3.55}$$

If we consider the Reissner-Nordstrom metric, then on the LHS of condition (3.55), the only non-zero Christoffel symbol tensor component would be Γ^r_{rr} and hence,

$$u'^r \Gamma^r_{rr} \sim \frac{3m}{2q^2}. \tag{3.56}$$

For Reissner-Nordstrom metric, the Γ^r_{rr} is given by :

$$\Gamma^r_{rr} = \frac{\frac{r_s}{2r} + \frac{r_Q^2}{r^2}}{r_s + \frac{r_Q^2}{r} - r}. \tag{3.57}$$

So, to find out the practical cases where the condition (3.56) holds, we need to first analyze the value of the Christoffel symbol component Γ^r_{rr} around typical black holes. The value of the quantity $\frac{3m}{2q^2}$, which is actually $\frac{6\pi\epsilon_0 c^3 m}{q^2}$ in S.I. units, on the RHS of condition (3.56), is of the order of 10^{26} for proton. We have already calculated and used the similar quantity in condition (3.35). There would be a limitation on the radial component of the four velocity $|u'^r|$, as it can not exceed the speed of light in vacuum i.e. c. Due to this limitation on $|u'^r|$, the order of the value of Γ^r_{rr} must be higher than 10^{18} for satisfying the condition for the case of proton.

Note that, in the expression of Γ^r_{rr} given in the Eqn.(3.57), for any charged particle or charged compact object orbiting around a typical Reissner-Norstdorm black hole, the radial distance r must be greater than the outer horizon, which is given by $r_+ = \frac{1}{2}(r_s + \sqrt{r_s^2 - 4r_Q^2})$.
For stellar mass black holes, the value of Γ^r_{rr} would decrease with increasing r and the same happens for supermassive black holes. But, for primordial black holes (PBHs), specially the ones having mass less than $10^{20}\,kg$, exactly the opposite happens. For instance, a PBH with mass $\sim 10^{20}\,kg$

would have Schwarzschild length scale $\sim 10^{-7}\,m$, and thus for any charged particle revolving around the PBH at a radial distance not exceeding $10^{-3.5}\,m$, the value of Γ^r_{rr} would be less than unity and this would increase with the decrease of the radial distance. Therefore, for systems where relativistic charged particles revolve around charged PBHs of sufficiently smaller masses, created in early Universe, the condition (3.56) could be easily satisfied resulting in significance of the term a^μ_5. In those cases even when the speeds of the charged particles is about $10^2\,ms^{-1}$ (while it is expected to be near the speed of light for smaller particles like proton), it would also not be difficult for the system to satisfy the condition (3.56).

The significance of the term $a^\mu_{int\,1}$: The ratio of the interaction term $a^\mu_{int\,1}$ to the gravitational radiation reaction term a^μ_1 is :

$$\frac{a^\mu_{int\,1}}{a^\mu_1} = \frac{-\frac{1}{2}\frac{q}{m}F^\mu_\nu u^\nu u^\alpha u^\beta h^R_{\alpha\beta}}{-\xi_1^2(\delta^\mu_\eta + u^\mu u_\eta)\Delta\Gamma^\eta_{\alpha\beta}u'^\alpha u'^\beta}\,. \tag{3.58}$$

So, for $a^\mu_{int\,1} \sim a^\mu_1$, the requirement is :

$$\frac{1}{2}\xi_1^2\frac{q}{m}F^\mu_\nu u^\nu u'^\alpha u'^\beta h^R_{\alpha\beta} \sim \xi_1^2 u^\mu u_\eta \Delta\Gamma^\eta_{\alpha\beta}u'^\alpha u'^\beta \tag{3.59}$$

Using the first part of the expression of $\Delta\Gamma^\eta_{\alpha\beta}$ given in the equation 3.16, in the above condition 3.59, we have :

$$\frac{q}{m}F^\mu_\eta u^\eta u'^\alpha u'^\beta h^R_{\alpha\beta} \sim u^\mu u_\eta h^{R\,\eta\sigma}(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta})u'^\alpha u'^\beta \tag{3.60}$$

As we did in previous cases, writing the above condition 3.60 in coordinate-wise manner for $\alpha, \beta = r$, we have :

$$\frac{q}{m}F^\mu_\eta u^\eta h^R_{rr} \sim u^\mu u_\eta h^{R\,\eta\sigma}(\partial_r g_{\sigma r} + \partial_r g_{\sigma r} - \partial_\sigma g_{rr}) \tag{3.61}$$

For Reissner-Nordstrom metric, the above condition 3.61 can be further simplified to :

$$\frac{q}{m}F^\mu_\eta u^\eta h^R_{rr} \sim u^\mu u^\eta h^R_{\eta r}(g^{rr}\partial_r g_{rr}) \tag{3.62}$$

If we expand the repeated index η on both sides of the condition 3.62 ; then again following the coordinate-wise manner and chosing the case of $\eta = r$ only, it reduces to :

$$\frac{q}{m}F^\mu_r \sim u^\mu(g^{rr}\partial_r g_{rr}) \tag{3.63}$$

For the Reissner-Nordstrom metric, we have already evaluated the quantity $g^{rr}\partial_r g_{rr}$ in the previous sub-section with $g_{rr} = (1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2})^{-1}$. Inserting the expression of $g^{rr}\partial_r g_{rr}$ in the above condition 3.63, we obtain :

$$\frac{q}{m}F^\mu_r \sim u^\mu \frac{\frac{1}{r}\left(\frac{r_s}{r} - \frac{2r_Q^2}{r^2}\right)}{\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)}\,. \tag{3.64}$$

We have already discussed about the quantity $\frac{\left(\frac{r_s}{r}-\frac{2r_Q^2}{r^2}\right)}{\left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right)}$ in the previous subsection, while discussing the condition of significance of the term a_5^μ. For the motion of a charged particle or a compact object near a Reissner-Nordstrom black hole, if we consider $r \sim r_s$, r_Q, the the above quantity is ~ 1. In that case, the above condition 3.64 simplifies to :

$$\frac{q}{m}F_r^\mu \sim \frac{1}{r}u^\mu \,. \tag{3.65}$$

In S.I. system of units we write this condition 3.65 as :

$$\frac{q}{4\pi\epsilon_0 c^3 m}F_r^\mu \sim \frac{1}{r}u^\mu \,. \tag{3.66}$$

For protons, the value of the quantity $\frac{q}{4\pi\epsilon_0 c^3 m}$ is $\approx 3.19\times 10^{-8}\,C^{-1}s$ and hence the condition 3.66 gives :

$$F_r^\mu \sim (3.13\times 10^7\,Cs^{-1})\frac{1}{r}u^\mu \,. \tag{3.67}$$

So, for protons or charged particles of similar category, it is expected that the above condition 3.67 is satisfied, if the charged particles move in the vicinity of a stellar-mass black hole with sufficient speed. Although, till now there is not probably any distinct observational evidence of electrically charged black holes, yet there can be external electromagnetic field, generated from any other source, near some astrophysical black holes and if this external electromagnetic field is sufficiently strong to satisfy the condition 3.67, then the interaction term $a_{int\,1}^\mu$ will be significant.

The significance of the term $a_{int\,2}^\mu$: From the ratio :

$$\frac{a_{int\,2}^\mu}{a_1^\mu} = \frac{-\frac{q}{m}(g^{\mu\nu}+u^\mu u^\nu)h^R_{\nu\alpha}F^\alpha_{\ \beta}u^\beta}{-\xi_1^2(\delta^\mu_\eta + u^\mu u_\eta)\Delta\Gamma^\eta_{\alpha\beta}u'^\alpha u'^\beta}\,, \tag{3.68}$$

we can say that one of the conditions for $a_{int\,2}^\mu \sim a_1^\mu$ is :

$$\frac{q}{m}h^{R\,\mu}_{\alpha}F^\alpha_{\ \beta}u^\beta \sim \Delta\Gamma^\mu_{\alpha\beta}u^\alpha u^\beta \tag{3.69}$$

Using the first part of the expression of $\Delta\Gamma^\mu_{\alpha\beta}$ given in the equation 3.16, in the above condition 3.69, we have :

$$\frac{q}{m}h^{R\,\mu\alpha}F_{\alpha\beta}u^\beta \sim h^{R\,\mu\sigma}(\partial_\alpha g_{\sigma\beta}+\partial_\beta g_{\sigma\alpha}-\partial_\sigma g_{\alpha\beta})u^\alpha u^\beta \,. \tag{3.70}$$

Now similar to the previous cases, considering the above condition 3.70 in coordinate-wise manner for $\alpha, \beta = r$, we get :

$$\frac{q}{m}h^{R\,\mu r}F_{rr} \sim h^{R\,\mu\sigma}(\partial_r g_{\sigma r}+\partial_r g_{\sigma r}-\partial_\sigma g_{rr})u^r \,. \tag{3.71}$$

For diagonal metrics like Reissner-Nordstrom metric, the above condition 3.71 will be further simplified to :

$$\frac{q}{m}h^{R\,\mu r}F_{rr} \sim h^{R\,\mu r}u^r\partial_r g_{rr} \tag{3.72}$$

or,

$$\frac{q}{m}F_{rr} \sim u^r \partial_r g_{rr} \,. \tag{3.73}$$

So, for Reissner-Nordstrom metric, the above condition 3.73 becomes :

$$\frac{q}{m}F_{rr} \sim u^r \frac{\frac{1}{r}\left(\frac{r_s}{r} - \frac{2r_Q^2}{r^2}\right)}{\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^2} \,. \tag{3.74}$$

For charged particles or charged compact objects moving near the black hole in such a way that $r \sim r_s,\, r_Q$, the quantity $\left(\frac{r_s}{r} - \frac{2r_Q^2}{r^2}\right)/\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^2$ in the R.H.S. of the above condition 3.74 is ~ 1. Then in that case, the above condition simplifies to :

$$\frac{q}{m}F_{rr} \sim \frac{u^r}{r} \,, \tag{3.75}$$

or in S.I. system of units, it can be written as :

$$\frac{q}{4\pi\epsilon_0 c^3 m}F_{rr} \sim \frac{u^r}{r} \,. \tag{3.76}$$

Similarly as before, substituting the value of the quantity $\frac{q}{4\pi\epsilon_0 c^3 m}$ for proton, we write the condition 3.76 as :

$$F_{rr} \sim (3.13 \times 10^7 \, Cs^{-1})\frac{u^r}{r} \,, \tag{3.77}$$

which is quite similar to the previous case of the condition 3.67. So, for the same cases, as discussed in the prvious subsection, this condition too would be satisfied.

3.6 Conclusion and Discussion :

In this chapter, we have shown that coexistence of metric perturbations and electromagnetic self-force can lead to an effect, in the motion of charged particles in curved space-time, which does not exist when any one among these two is absent. In most of the physical situations, the metric perturbation is the gravitational radiation emitted from the charged particle itself, which causes the gravitational self-force. The perturbative terms of the electromagnetic self-force, which we have derived, are different from the interaction terms of electromagnetic and gravitational self-forces, given in the work of P. Zimmerman and E. Poisson [98].

We have analyzed different conditions for which these perturbative terms generated from the electromagnetic self-force due to its perturbation by the gravitational radiation would be significant in comparison with the gravitational self-force. We have also analyzed the conditions of significance of the interaction terms of electromagnetic and gravitational self-forces in comparison with the gravitational self-force.

It is interesting to find that there are astrophysical phenomena and cosmological cases where these perturbative terms can play a significant role. The physical interpretation of these perturbations to the electromagnetic self-force by the gravitational radiation can be understood as the fact that, when electromagnetic self-force is acting in curved space-time, where there is gravitational wave emission from the system, then the electromagnetic wave produced due to the motion of the charged particle has to traverse through the ripples in the curved space-time due to the grav-

itational radiation. However, in the absence of the gravitational radiation, the electromagnetic wave propagates through the curved space-time but it does not face ripples in space-time. It is this difference with the case where gravitational radiation is present, that manifests in the form of these perturbative terms generated in the equation of motion of the charged particle. It is important to note that not taking into account these perturbative terms in the specified astrophysical or cosmological phenomena involving relativistic charged particles or compact objects, and considering only the gravitational radiation reaction can lead to incorrect estimation of their motions.
In this way, we have also demonstrated that it can be misleading if we estimate or compare magnitude of terms in the equation-of-motion of the charged particle, only in terms of q and m. The other physical quantities present in the terms also matter. They can play a significant role in determining the overall order of certain term. We have not dicussed the case where the source of gravitational radiation perturbing the system is external. But, from the study in this chapter we can say, that for a suitable external source of gravitational radiation too, the perturbative terms of the electromagnetic self-force can be significant in comparison with the gravitational self-force.

3.7 APPENDIX-1 : Orthogonality of the radiation reaction terms with four-velocity :

3.7.1 The orthogonality properties of different terms in a^μ :

Now, we test the orthogonality of the radiation reaction term a^μ. We just write the expression of a^μ from Eqn.(5.3) in the following way :

$$a^\mu = \frac{Du^\mu}{d\tau} - \frac{q}{m}F^\mu_{\ \nu}u^\nu + \frac{2q^2}{3m}\Big(\frac{D^2u^\mu}{d\tau^2} + u^\mu u_\nu \frac{D^2u^\nu}{d\tau^2}\Big) + \\ \frac{q^2}{3m}(R^\mu_\lambda u^\lambda + R^\nu_\lambda u^\lambda u^\mu u_\nu) + \frac{2q^2}{m} f^{\mu\nu}_{Tail} u_\nu. \tag{3.78}$$

It is already known that the part $\frac{Du^\mu}{d\tau}$ is perpendicular to the four-velocity u^μ [100] i.e.

$$\frac{Du^\mu}{d\tau}u_\mu = \Big(\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\Big)u_\mu = 0. \tag{3.79}$$

Next, we test the orthogonality with the electromagnetic radiation reaction term $\frac{2q^2}{3m}(g^\mu_\nu + u^\mu u_\nu)\frac{D^2u^\nu}{d\tau^2}$, using the relation $u^\alpha u_\alpha = -1$:

$$\Big(g^\mu_\nu + u^\mu u_\nu\Big)\frac{D^2u^\nu}{d\tau^2}u_\mu = \Big(g^\mu_\nu u_\mu + u^\mu u_\mu u_\nu\Big)\frac{D^2u^\nu}{d\tau^2} = (u_\nu + (-1)u_\nu)\frac{D^2u^\nu}{d\tau^2} = 0. \tag{3.80}$$

The orthogonality of the Lorentz force term $\frac{q}{m}F^\mu_{\ \nu}u^\nu$ results from the anti-symmetry of the field strength tensor under the exchange of the space-time indices. Then, we test the orthogonality of the tail term $\frac{2q^2}{m}f^{\mu\nu}_{Tail}u_\nu$. The $f^{\mu\nu}_{Tail}$ in the tail term is the 'tail integral' given by [101, 106]:

$$f^{\mu\nu}_{Tail} = \int_{-\infty}^{\tau-0^+} D^{[\mu}G^{\nu]}_{+\lambda''}(z(\tau), z(\tau''))u^{\lambda''}d\tau''. \tag{3.81}$$

Where $G^{\mu}_{+\lambda}$ is the retarded Green's function associated with the vector potential of the electromagnetic field. Hence, contracting the term $f^{\mu\nu}_{Tail}u_{\nu}$ with u_{μ} we have:

$$\begin{aligned} f^{\mu\nu}_{Tail}u_{\nu}u_{\mu} = u_{\nu}u_{\mu}\int_{-\infty}^{\tau-0^{+}} D^{[\mu}G^{\nu]}_{+\lambda''}(z(\tau),z(\tau''))u^{\lambda''}d\tau'' \\ = u_{\nu}u_{\mu}\int_{-\infty}^{\tau-0^{+}} (D^{\mu}G^{\nu}_{+\lambda''} - D^{\nu}G^{\mu}_{+\lambda''})(z(\tau),z(\tau''))u^{\lambda''}d\tau'' \\ = u_{\nu}u_{\mu}\int_{-\infty}^{\tau-0^{+}} D^{\mu}G^{\nu}_{+\lambda''}(z(\tau),z(\tau''))u^{\lambda''}d\tau'' \\ -u_{\nu}u_{\mu}\int_{-\infty}^{\tau-0^{+}} D^{\nu}G^{\mu}_{+\lambda''}(z(\tau),z(\tau''))u^{\lambda''}d\tau''. \end{aligned} \tag{3.82}$$

In this case also, in the two parts on the RHS of the above Eqn.(3.82), the indices μ and ν are repeated and in the similar way, as can be done for the previous case of Lorentz force, in this case also we interchange the indices ($\mu \leftrightarrow \nu$) for the first term on the RHS of Eqn.(3.82) and obtain :

$$\begin{aligned} f^{\mu\nu}_{Tail}u_{\nu}u_{\mu} = u_{\nu}u_{\mu}\int_{-\infty}^{\tau-0^{+}} D^{\mu}G^{\nu}_{+\lambda''}(z(\tau),z(\tau''))u^{\lambda''}d\tau'' \\ -u_{\nu}u_{\mu}\int_{-\infty}^{\tau-0^{+}} D^{\nu}G^{\mu}_{+\lambda''}(z(\tau),z(\tau''))u^{\lambda''}d\tau'' \\ = u_{\mu}u_{\nu}\int_{-\infty}^{\tau-0^{+}} D^{\nu}G^{\mu}_{+\lambda''}(z(\tau),z(\tau''))u^{\lambda''}d\tau'' \\ -u_{\nu}u_{\mu}\int_{-\infty}^{\tau-0^{+}} D^{\nu}G^{\mu}_{+\lambda''}(z(\tau),z(\tau''))u^{\lambda''}d\tau'' = 0. \end{aligned} \tag{3.83}$$

Hence, the overall radiation reaction term a^{μ} is orthogonal to the four-velocity u_{μ} : $a^{\mu}u_{\mu} = 0$. In the next sub-section we shall show the utility of this orthogonality property.

3.7.2 Utility of the Orthogonality Property of the reaction a^{μ} with the four-velocity :

As we have the orthogonality property of the overall radiation reaction, we can use it to have constraints or relations between different coefficients and terms present in the reaction. For doing this, we contract the radiation reaction a^{μ} with unperturbed four-velocity u_{μ} in the Eqn.(5.6).

Thus we get :

$$\begin{aligned}
&\left\{\frac{d^2\tau'}{d\tau^2}\frac{d\tau}{d\tau'}\frac{dx^\mu}{d\tau}+(-\Delta\Gamma^\mu_{\nu\rho})\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\right\}u_\mu+\left(\frac{d\tau'}{d\tau}\right)\frac{q}{m}\left(F'^{\mu\nu}-\frac{d\tau}{d\tau'}F^{\mu\nu}\right)u_\nu u_\mu+\\
&\frac{2q^2}{m}\left(\frac{d\tau'}{d\tau}\right)\left(f'^{\mu\nu}_{Tail}-\frac{d\tau}{d\tau'}f^{\mu\nu}_{Tail}\right)u_\nu u_\mu+\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{d^3x^\eta}{d\tau'^3}\left\{\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)u_\mu-\frac{d\tau'}{d\tau}(0)\right\}+\\
&\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\alpha}{d\tau'}\frac{dx^\beta}{d\tau'}\frac{dx^\gamma}{d\tau'}\left\{\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)\partial_\gamma\Gamma'^\eta_{\alpha\beta}u_\mu-\frac{d\tau'}{d\tau}(0)\right\}+\\
&\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\alpha}{d\tau'}\frac{d^2x^\beta}{d\tau'^2}\left\{\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)3\Gamma'^\eta_{\alpha\beta}u_\mu-\frac{d\tau'}{d\tau}(0)\right\}+\\
&\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\alpha}{d\tau'}\frac{dx^\rho}{d\tau'}\frac{dx^\sigma}{d\tau'}\left\{\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)\Gamma'^\eta_{\alpha\beta}\Gamma'^\beta_{\rho\sigma}u_\mu-\frac{d\tau'}{d\tau}(0)\right\}-\\
&\frac{2q^2}{3m}\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)\left\{3\Gamma^\eta_{\alpha\beta}\frac{d\tau'}{d\tau}\frac{d^2\tau'}{d\tau^2}\left(\frac{dx^\alpha}{d\tau'}\frac{dx^\beta}{d\tau'}\right)+\frac{d^3\tau'}{d\tau^3}\frac{d\tau'}{d\tau}\frac{dx^\eta}{d\tau'}+3\frac{d\tau'}{d\tau}\frac{d^2\tau'}{d\tau^2}\frac{d^2x^\eta}{d\tau'^2}\right\}u_\mu=a^\mu u_\mu.
\end{aligned}\tag{3.84}$$

We have already shown that $F^{\mu\nu}u_\nu u_\mu=0$ and $f^{\mu\nu}_{Tail}u_\mu u_\nu=0$ in the previous sub-section. The similar results are also valid for the perturbed external Lorentz force and the perturbed tail term : $F'^{\mu\nu}u_\nu u_\mu=0$ and $f'^{\mu\nu}_{Tail}u_\mu u_\nu=0$. Using those results and the fact that $\left(g^\mu_\eta+\frac{dx^\mu}{d\tau}\frac{dx_\eta}{d\tau}\right)u_\mu=0$, in the above Eqn.(3.84), we can write it in the following way :

$$\begin{aligned}
&\left\{\frac{d^2\tau'}{d\tau^2}\frac{d\tau}{d\tau'}(-1)+(-\Delta\Gamma^\mu_{\nu\rho})u_\mu\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\right\}+\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{d^3x^\eta}{d\tau'^3}\left\{\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)u_\mu\right\}+\\
&\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\alpha}{d\tau'}\frac{dx^\beta}{d\tau'}\frac{dx^\gamma}{d\tau'}\left\{\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)\partial_\gamma\Gamma'^\eta_{\alpha\beta}u_\mu\right\}+\\
&\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\alpha}{d\tau'}\frac{d^2x^\beta}{d\tau'^2}\left\{\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)3\Gamma'^\eta_{\alpha\beta}u_\mu\right\}+\\
&\frac{2q^2}{3m}\left(\frac{d\tau'}{d\tau}\right)^2\frac{dx^\alpha}{d\tau'}\frac{dx^\rho}{d\tau'}\frac{dx^\sigma}{d\tau'}\left\{\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)\Gamma'^\eta_{\alpha\beta}\Gamma'^\beta_{\rho\sigma}u_\mu\right\}-\\
&\frac{2q^2}{3m}\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)\left\{3\Gamma^\eta_{\alpha\beta}\frac{d\tau'}{d\tau}\frac{d^2\tau'}{d\tau^2}\left(\frac{dx^\alpha}{d\tau'}\frac{dx^\beta}{d\tau'}\right)+\frac{d^3\tau'}{d\tau^3}\frac{d\tau'}{d\tau}\frac{dx^\eta}{d\tau'}+3\frac{d\tau'}{d\tau}\frac{d^2\tau'}{d\tau^2}\frac{d^2x^\eta}{d\tau'^2}\right\}u_\mu=0.
\end{aligned}\tag{3.85}$$

Now, we simplify the expression $\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)u_\mu$:

$$\begin{aligned}
\left(g'^\mu_\eta+\frac{dx^\mu}{d\tau'}\frac{dx_\eta}{d\tau'}\right)u_\mu&=(g^\mu_\eta+h^{R\,\mu}_\eta)u_\mu+\left(\frac{d\tau}{d\tau'}\right)^2\frac{dx^\mu}{d\tau}\frac{dx_\eta}{d\tau}u_\mu\\
&=(g^\mu_\eta u_\mu+h^{R\,\mu}_\eta u_\mu)+\left(\frac{d\tau}{d\tau'}\right)^2u_\eta(u^\mu u_\mu)\\
=(u_\eta+h^{R\,\mu}_\eta u_\mu)+\left(\frac{d\tau}{d\tau'}\right)^2u_\eta(-1)&=u_\eta\left(1-\left(\frac{d\tau}{d\tau'}\right)^2\right)+h^{R\,\mu}_\eta u_\mu.
\end{aligned}\tag{3.86}$$

Using this above expression and simplifying the Eqn(3.85), we obtain :

$$
\begin{aligned}
-\left\{\frac{d^2\tau'}{d\tau^2}\frac{d\tau}{d\tau'} + \Delta\Gamma^{\mu}_{\nu\rho}u_{\mu}u^{\nu}u^{\rho}\right\} = -\left\{u_{\eta}\Big(1-\Big(\frac{d\tau}{d\tau'}\Big)^2\Big) + h_{\eta}^{R\,\mu}u_{\mu}\right\}\frac{2q^2}{3m} \\
\Big[\Big(\frac{d\tau'}{d\tau}\Big)^2\Big(\frac{d^3x^{\eta}}{d\tau'^3} + u'^{\alpha}u'^{\beta}u'^{\gamma}\partial_{\gamma}\Gamma'^{\eta}_{\alpha\beta} + 3u'^{\alpha}\frac{du'^{\beta}}{d\tau'}\Gamma'^{\eta}_{\alpha\beta} + u'^{\alpha}u'^{\rho}u'^{\sigma}\Gamma'^{\eta}_{\alpha\beta}\Gamma'^{\beta}_{\rho\sigma}\Big) \\
-\Big(3\Gamma^{\eta}_{\alpha\beta}\frac{d\tau'}{d\tau}\frac{d^2\tau'}{d\tau^2}\Big(\frac{dx^{\alpha}}{d\tau'}\frac{dx^{\beta}}{d\tau'}\Big) + \frac{d^3\tau'}{d\tau^3}\frac{d\tau'}{d\tau}\frac{dx^{\eta}}{d\tau'} + 3\frac{d\tau'}{d\tau}\frac{d^2\tau'}{d\tau^2}\frac{d^2x^{\eta}}{d\tau'^2}\Big)\Big].
\end{aligned}
\tag{3.87}
$$

Now, using the above Eqn.(3.87) we can substitute the terms $\Big(3\Gamma^{\eta}_{\alpha\beta}\frac{d\tau'}{d\tau}\frac{d^2\tau'}{d\tau^2}\Big(\frac{dx^{\alpha}}{d\tau'}\frac{dx^{\beta}}{d\tau'}\Big) + \frac{d^3\tau'}{d\tau^3}\frac{d\tau'}{d\tau}\frac{dx^{\eta}}{d\tau'} + 3\frac{d\tau'}{d\tau}\frac{d^2\tau'}{d\tau^2}\frac{d^2x^{\eta}}{d\tau'^2}\Big)$ in the Eqn.(5.7) in terms of $g_{\mu\nu}$, $h_{\mu\nu}, \Gamma^{\mu}_{\nu\rho}, \Delta\Gamma^{\mu}_{\nu\rho}, u_{\mu}$ etc.

3.8 APPENDIX-2 : The reason for neglecting the term containing the Ricci-tensor :

As there is a background electromagnetic field in this case due to the external electromagnetic field and also due to the electromagnetic field emitted by the charged particle, so the term $\frac{q^2}{3m}(R^{\mu}_{\lambda}u^{\lambda} + R^{\nu}_{\lambda}u^{\lambda}u^{\mu}u_{\nu})$ is in general non-zero. But, we can have an idea of its order by some analysis. We show this here.

As the electromagnetic field is a kind of radiation, with EoS-parameter w= 1/3, the Ricci-scalar due to it vanishes and this can be shown simply from the Einstein's equation viz. the Einstein-Maxwell's equation here.

We start from the Einstein-Maxwell's equation or the Einstein's equation with electromagnetic stress-energy tensor as the source :

$$
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4\mu_o}(F_{\mu}{}^{\eta}F_{\nu\eta} - \frac{1}{4}g_{\mu\nu}F^2)\,,
\tag{3.88}
$$

where μ_0 is the permeability in vacuum and $F_{\mu\nu}$ is the electromagnetic field tensor. Contracting both sides of the above equation 3.88 with $g^{\mu\nu}$, we get :

$$
\begin{aligned}
R_{\mu\nu}g^{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\mu\nu}R = \\
\frac{8\pi G}{c^4\mu_o}(F_{\mu}{}^{\eta}F_{\nu\eta}g^{\mu\nu} - \frac{1}{4}g_{\mu\nu}g^{\mu\nu}F^2)\,, \\
\Rightarrow R - \frac{1}{2}(4)R = \frac{8\pi G}{c^4\mu_o}(F_{\mu}{}^{\eta}F^{\mu}_{\eta} - \frac{1}{4}(4)F^2) \\
\Rightarrow -R = \frac{8\pi G}{c^4\mu_o}(F^2 - F^2) = 0
\end{aligned}
\tag{3.89}
$$

So, we have shown that for electromagnetic field the Ricci-scalar R is zero (0). Now, we substitute R = 0 in the equation 3.88 and get :

$$
R_{\mu\nu} = \frac{8\pi G}{c^4\mu_o}(F_{\mu}{}^{\eta}F_{\nu\eta} - \frac{1}{4}g_{\mu\nu}F^2)\,,
\tag{3.90}
$$

Or, in mixed-tensorial form,

$$R^{\mu}_{\nu} = \frac{8\pi G}{c^4\mu_o}(F^{\mu\eta}F_{\nu\eta} - \frac{1}{4}g^{\mu}_{\nu}F^2)\,. \tag{3.91}$$

The term containing Ricci-tensors, appearing in the equation of motion of a charged particle in curved space-time, is $\frac{q^2}{3m}(R^{\mu}_{\lambda}u^{\lambda} + R^{\nu}_{\lambda}u^{\lambda}u^{\mu}u_{\nu})$. We first evaluate the first part $R^{\mu}_{\lambda}u^{\lambda}$:

$$\begin{aligned} R^{\mu}_{\lambda}u^{\lambda} &= \frac{8\pi G}{c^4\mu_o}(F^{\mu\eta}F_{\lambda\eta}u^{\lambda} - \frac{1}{4}g^{\mu}_{\lambda}u^{\lambda}F^2) \\ &= \frac{8\pi G}{c^4\mu_o}(F^{\mu\eta}F_{\lambda\eta}u^{\lambda} - \frac{1}{4}u^{\mu}F^2)\,. \end{aligned} \tag{3.92}$$

Then, we evaluate the second part :

$$R^{\nu}_{\lambda}u^{\lambda}u^{\mu}u_{\nu} = \frac{8\pi G}{c^4\mu_o}(F^{\nu\eta}F_{\lambda\eta}u^{\lambda} - \frac{1}{4}g^{\nu}_{\lambda}u^{\lambda}F^2)u^{\mu}u_{\nu} \tag{3.93}$$

$$\Rightarrow R^{\nu}_{\lambda}u^{\lambda}u^{\mu}u_{\nu} = \frac{8\pi G}{c^4\mu_o}(F^{\nu\eta}F_{\lambda\eta}u^{\lambda}u^{\mu}u_{\nu} + \frac{1}{4}u^{\mu}F^2)\,. \tag{3.94}$$

Hence,

$$\begin{aligned} R^{\mu}_{\lambda}u^{\lambda} + R^{\nu}_{\lambda}u^{\lambda}u^{\mu}u_{\nu} &= \\ \frac{8\pi G}{c^4\mu_o}(F^{\mu\eta}F_{\lambda\eta}u^{\lambda} - \frac{1}{4}u^{\mu}F^2 + F^{\nu\eta}F_{\lambda\eta}u^{\lambda}u^{\mu}u_{\nu} + \frac{1}{4}u^{\mu}F^2) \\ = \frac{8\pi G}{c^4\mu_o}(F^{\mu\eta} + F^{\nu\eta}u^{\mu}u_{\nu})F_{\lambda\eta}u^{\lambda} = \frac{8\pi G}{c^4\mu_o}(\delta^{\mu}_{\nu} + u^{\mu}u_{\nu})F^{\nu\eta}F_{\lambda\eta}u^{\lambda} \end{aligned} \tag{3.95}$$

Hence,

$$\begin{aligned} &\frac{q^2}{12\pi\epsilon_0 c^3 m}(R^{\mu}_{\lambda}u^{\lambda} + R^{\nu}_{\lambda}u^{\lambda}u^{\mu}u_{\nu}) \\ &= \frac{q^2}{12\pi\epsilon_0 c^3 m}\frac{8\pi G}{c^4\mu_o}(\delta^{\mu}_{\nu} + u^{\mu}u_{\nu})F^{\nu\eta}F_{\lambda\eta}u^{\lambda}\,, \end{aligned} \tag{3.96}$$

where, we have written the coefficient $q^2/3m$, as written in general calculations of our manuscript, in S.I. units as $\frac{q^2}{12\pi\epsilon_0 c^3 m}$; ϵ_0 being the permittivity of vacuum. It is to be noted that the term $qF_{\lambda\eta}u^{\lambda}$ within the R.H.S.(right hand side) of the equation 3.96 is just the Lorentz-force due to the electromagnetic field $F_{\lambda\eta}$. Among the rest of the terms $(\delta^{\mu}_{\nu} + u^{\mu}u_{\nu})$ has the order ~ 1. Now, the rest coefficient is :

$$\frac{8\pi G}{12\pi\epsilon_0\mu_o c^7}\frac{q}{m}F^{\nu\eta} = \frac{2G}{3c^5}\frac{q}{m}F^{\nu\eta}\,, \tag{3.97}$$

as we know that $\epsilon_0\mu_0 = c^{-2}$. Evaluating the numerical value of the quantity $\frac{2G}{3c^5}$, we get :

$$\frac{2G}{3c^5} \approx 1.83\times 10^{-53}\,kg^{-1}m^{-2}s^3\,. \tag{3.98}$$

Therefore, there can hardly be any astrophysical and cosmological cases, which we have discussed in this chapter, having charge-to-mass ratio so high, that it can compensate the very much small factor 10^{-53} . On the other hand it is hardly possible to find any astrophysical or cosmological case where the external electric or magnetic field is so huge that the electromagnetic-field can compensate the factor 10^{-53}.

Generally, the astrophysical compact objects like neutron stars, white dwarfs and black holes

can have very high magnetic fields $\sim 10^{17}\,G$ or $10^{13}\,T$, associated with them. On the other hand, we have already stated that theoretically maximum-possible value of electric fields, that can be produced by charged neutron stars near them, is $\sim 10^{21}V/m$. So, it is quite clear that even these very high magnetic fields and electric fields are also insufficient to make this term $\frac{q^2}{3m}(R^{\mu}_{\lambda}u^{\lambda} + R^{\nu}_{\lambda}u^{\lambda}u^{\mu}u_{\nu})$ significant, overcoming the extremely small factor 10^{-53}. Furthermore in the present work discussed in this chapter, we would have to deal with the perturbations of this term. So, no doubt that if this term is of so small order, then its perturbations created due to the metric fluctuations generated from the charged particle or compact object, would be even smaller. So, it is now clear that the perturbative correction terms generated from this term $\frac{q^2}{3m}(R^{\mu}_{\lambda}u^{\lambda} + R^{\nu}_{\lambda}u^{\lambda}u^{\mu}u_{\nu})$ can be neglected.
On the other hand, the Abraham-Lorentz-Dirac term $\frac{2q^2}{3m}(\delta^{\mu}_{\nu} + u'^{\mu}u'_{\nu})\frac{D^2u'^{\nu}}{d\tau^2}$ can not be expressed in the style of equation 3.96, so the same can not be said for it.
For this reason, we are neglecting the perturbations generated from this term containing Ricci tensor in our work.

3.9 APPENDIX-3 : Certain explanations on comparing different parts within the correction terms, while investigating the significance of the correction terms :

In this section, we clarify certain issues regarding the comparison of different parts of the corrections terms, given in equations 3.21 to 5.21, separately with the gravitational self-force term. In the numerator of the R.H.S. of the equation 5.22, it may seem that we have not compared the term $\frac{2q^2}{m}\xi_1^2 u'^{\alpha}\frac{du'^{\beta}}{d\tau'}\Delta\Gamma^{\mu}_{\alpha\beta}$. But, actually if we compare both the terms together viz. $\frac{2q^2}{m}\xi_1^2 u'^{\alpha}\frac{du'^{\beta}}{d\tau'}(\delta^{\mu}_{\eta} + u'^{\mu}u'_{\eta})\Delta\Gamma^{\eta}_{\alpha\beta}$, in the numerator with that of the denominator viz. $\xi_1^2(\delta^{\mu}_{\eta} + u^{\mu}u_{\eta})$, then the resultant condition will be same as that, which is shown in the equation 5.24. [It is to be noted that $u'^{\mu}u'_{\eta} \approx u^{\mu}u_{\eta}$]. So, we do not need to separately compare the terms $\frac{2q^2}{m}\xi_1^2 u'^{\alpha}\frac{du'^{\beta}}{d\tau'}u'^{\mu}u'_{\eta}\Delta\Gamma^{\eta}_{\alpha\beta}$ and $\frac{2q^2}{m}\xi_1^2 u'^{\alpha}\frac{du'^{\beta}}{d\tau'}\Delta\Gamma^{\mu}_{\alpha\beta}$ with that of the denominator. And, we have already stated that we have compared the term $\frac{2q^2}{m}\xi_1^2 u'^{\alpha}\frac{du'^{\beta}}{d\tau'}h^{R\,\mu}_{\eta}\Gamma^{\eta}_{\alpha\beta}$, starting from equation 3.38.
For the subsection 5.2 i.e. for comparison of the term a^{μ}_4 with a^{μ}_1, the similar argument is valid, as given above. If we compare the two terms : first and third ones together, in the numerator on the R.H.S. of the equation 3.42, with that of the denominator, then the ultimate result will be the same as that given in the equation 5.32. On the other hand, comparison of the rest term $\frac{2q^2}{3m}\xi_1^2 u'^{\alpha}u'^{\beta}u'^{\gamma}h^{R\,\mu}_{\eta}$ with the denominator yields such a condition, that may not be predicted to be satisfied in any certain astrophysical or cosmological case. That is why, we have not compared that one.
Similarly, in the subsection 5.3 i.e. for comparison of the term a^{μ}_5 with a^{μ}_1, same logic stands.

Chapter 4

Enhanced power of gravitational waves and rapid coalescence of black hole binaries through k-essence dark energy accretion

4.1 Introduction

After the first direct detection of gravitational waves from a merging binary of black holes by aLIGO [5], and subsequent series of detections from similar sources [6, 7, 8, 9], a new era in observational astronomy has begun. Besides binaries of compact objects in bounded orbits, there are various other mechanisms of production of gravitational waves from a wide varieties of sources, such as nearby fly-pass of two compact objects in unbounded orbits [12], gravitational collapse of sufficiently massive stars [122], cosmological phase transitions [123, 124], breaking of cosmic strings [125, 126], inflation and pre-heating [127, 128], dark sector interactions [129], etc. However, till date the observations by the aLIGO and VIRGO detectors have been carried out from mainly one type of sources, which are the binaries of compact objects, *viz.*, black holes and neutron stars. Gravitational wave observations have been used to estimate and constrain various astrophysical and cosmological parameters associated with the generation and propagation of these gravitational waves. Among these, some important ones worth mentioning are: (i) estimating the Hubble parameter [130, 131], (ii) constraining a large class of cosmological scalar-tensor theories [132, 133], (iii) constraining the mass of gravitons for bimetric-gravity theories [134], (iv) investigating the state of matter inside a neutron star [135], (v) constraining higher-dimensional theories [136], and there are several others. Attempts to constrain dark energy, responsible for the accelerated expansion of the late Universe [137], have been made indirectly using gravitational wave observations, either through the estimation of the Hubble parameter, or through constraining cosmological scalar-tensor theories.

Late time acceleration of expansion of the Universe is one of the most intriguing discoveries of recent times, which was directly confirmed from supernovae Ia observations in 1998 [138, 139] and was also supported by various indirect probes. Many theoretical approaches have been employed to explain the current cosmic acceleration. The component of the Universe providing the required negative pressure for driving this accelerated expansion is generically called 'dark energy' [140]. As normal matter (radiation, baryonic matter or cold dark matter) is gravitationally attractive, the standard lore is to assume the presence of a relativistic fluid which is repulsive in nature, as the dark energy candidate. The simplest candidate of dark energy is the cosmological constant Λ, which is mostly consistent with cosmological observations. However, it is plagued with conceptual problems, for example, fine-tuning and coincidence problems [141], which are theoretical in nature. With a hope to address these problems, cosmologists have proposed mechanisms where the role of dark energy is played by a completely different component of the Universe, which may have a variable equation-of-state parameter.

Many varieties of dark energy models have been proposed, theoretically studied and observationally constrained till now. There exist a wide class of scalar field models coupled to gravity. Among these, minimally coupled ones, called quintessence, in which cosmic acceleration is driven by the potential energy [142, 143], are known to alleviate some of the problems of the cosmological constant. Scalar fields, in which the cosmic acceleration is driven by the kinetic energy, called 'k-essence' [144, 145, 146], have also been studied, motivated from unification and quantum gravity scenarios. Such models may further yield a consistent picture of the complete evolution[1], starting from the early era inflation, the subsequent dark matter domination, and finally the late time acceleration [154, 155]. Other alternatives include random barotropic fluids with pre-determined forms of the equation-of-state parameter, such as the Chaplygin gas models [156], string theory motivated models [153, 157, 158] and braneworld models [159, 160]. There also exist approaches without requiring additional fields[161, 162, 163, 164].

A major difference between the scalar field and other fluid models of dark energy with the ΛCDM model (and other approaches not requiring additional fields) is that the former type of dark energy is subjected to accretion by the black holes present in the Universe. In fact, those back holes with surroundings containing insufficiently available other forms of matter-energy for accretion, would still accrete the scalar field dark energy, which is uniformly distributed throughout the Universe. Accretion of various types of dark energy by black holes has been a subject of theoretical interest for a considerable time [165, 166, 167, 168, 169]. On the basis of various works done till date, it is widely accepted that the mass of a black hole would increase due to steady spherical accretion if the equation-of-state parameter of the dark energy w_{DE} is > 1. On the other hand, accretion would result in mass loss of a black hole, for phantom type dark energy with $w_{DE} < 1$ [166] in the case of fixed background. However, it has been pointed out [167] that if one takes into account the backreaction which becomes large when the dark energy background becomes large (for the case of phantom dark energy this will eventually happen), the black hole mass can increase when accreting the dark energy with $w_{DE} < -1$. [2]

If dark energy exists in the Universe in a form which can be accreted by black holes, the result would not be limited to just the change of masses of the black holes. It is expected that other phenomena associated with the black holes would also be influenced. The evolution of binaries

[1] Recently, there exists an interesting debate on the impact of the Hubble tension [147, 148] on the viability of dark energy models. It is claimed that a large value of local determination of the Hubble parameter may severly constrain dark energy models [149, 150], though schemes for resolution of the Hubble tension have also been proposed using early dark energy (EDE) models [151]. Interestingly, the k-essence framework is also conducive to the EDE picture [152].

[2] If the background space-time is evolving, which is the case for the FLRW-Universe, then it has been shown that according to the notions of quasilocal mass and the generalized Misner-Sharp mass, the black hole mass increases due to accretion of phantom dark energy with $w_{DE} < -1$ [167]. It has been argued that the discrepancy of increasing quasilocal mass and apparent horizon of a black hole, due to accretion of a phantom fluid, with the decrease in mass of a black hole in fixed background space-time, is due to the reason that the quasilocal mass does not receive contributions from the negative pressure. In the present work, we do not need to consider quasilocal mass of the black holes, hence it is not required to consider the above mentioned effect.

formed with the black holes, the gravitational wave emitted from those binaries and coalescence of those binaries are some of the physical processes which get directly affected if the masses of the concerned black holes are changing continuously instead of being constant. Several works have been done exploring the effect of accretion of gas on compact binaries emitting gravitational waves and detectability of the imprints of this effect [170, 171, 172, 173, 174]. In the context of spherically symmetric accretion of dark energy [165], the efficacy of the above effects, in particular, whether the modified variation of gravitational energy of the binary system could be detectable via the rate of change of the orbital radius, has been a subject of debate [175, 176].

In the present work described in this chapter, our motivation is to explore the problem of associated modification of black hole binary parameters due to accretion in the context of a popular k-essence model of dark energy. Specifically, we consider a string theory [177] inspired low energy effective action framework containing a dilaton scalar field [178]. The resultant k-essence dark energy scenario [179] is compatible with cosmological observations [180]. Here we study spherical accretion of the k-essence dilatonic ghost condensate dark energy by black holes. This falls within a class of models known as 'ghost condensates' [181]. Considering binary formations of black holes in the early inspiral stage, we study main aspects of evolution of the orbits, due to continuous change of masses of the component black holes of such binaries, resulting from spherical accretion of the chosen model of k-essence dark energy. More specifically, we study the pace of shrinking of such an orbit and the average power of the gravitational wave emitted from the orbit in the course of its evolution, and perform quantitative comparisons of the differences with the case when the masses of component black holes are constant. We further investigate the modification in the time required to reach the coalescence stage of such a binary, in comparison with the constant mass case.

The chapter is organised as follows. In section 4.2, we study the growth of black hole mass due to accretion of the chosen model of k-essence dark energy in the late Universe. In section 4.3, we investigate the effect of growing masses of black holes on the evolution of binaries. We compute the rate of decrease of orbital radius after circularization of orbits, and the average power of the emitted gravitational waves. We compare these results with the case of binaries with black holes of constant masses without accretion. In section 4.4, we estimate the reduction in the time required for reaching coalescence-stage by such binaries. We present our concluding remarks in section 4.5.

4.2 Dark energy accretion by black holes in a k-essence model

K-essence scalar fields are the dynamical dark energy models where the acceleration is driven by kinetic term in the scalar field Lagrangian. Among many k-essence models, we choose a particular string-inspired ghost condensate model, called 'k-essence dilatonic ghost condensate', which can successfully describe the cosmological evolution, while simultaneously satisfies the necessary conditions of quantum stability and sound speed [179, 182]. This model has also found to be observationally consistent [180].

The condition on sound speed for any scalar field dark energy model is simply that the sound speed can not exceed the speed of light in vacuum (c) i.e. it can not have super-luminal speed. In this regard, it is worthwhile to mention that the sound speed makes an important difference between quintessence and k-essence models. While for standard quintessence models, with canonical

scalar fields, the sound speed is always equal to the speed of light ; for the k-essence models it is not so. This fact of having varying sound speeds through the cosmic evolution gives various ways of distinguishing different k-essence models from one-another and from the quintessence models [183]. In fact this difference of sound speed of k-essence models with the quintessence models is one of the main reasons, for which we have chosen a k-essence model for this work.
The action of k-essence scalar field φ, along with non-relativistic matter and radiation, can be generally written as [144] :

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R + \mathcal{L}(\varphi, X)\right\} + S_m\,, \tag{4.1}$$

where $\kappa = (8\pi G/3)^{1/2}$, R is the Ricci-scalar and $\mathcal{L}$ is a function of the k-essence scalar field φ and its kinetic energy $X = -(1/2)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$. S_m is the action contributed from the non-relativistic matter and radiation. In case of the specific model considered here, the Lagrangian density is given by [179, 182]:

$$\mathcal{L} = -X + e^{\kappa\lambda\varphi}\frac{X^2}{M^4}\,, \tag{4.2}$$

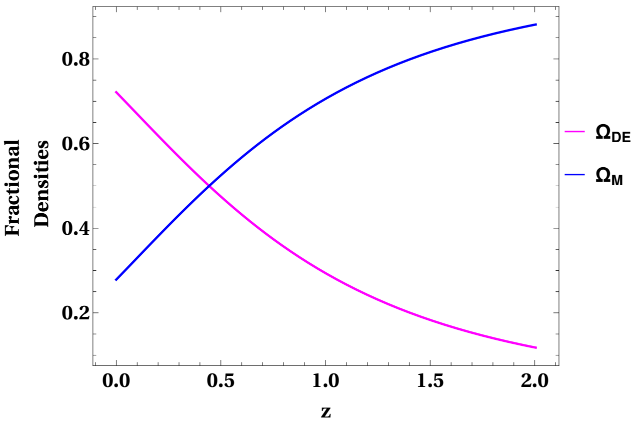

Figure 4.1: Evolution of the fractional densities of k-essence dark energy denoted by $\Omega_{DE} \equiv \Omega_\varphi$ and non-relativistic matter denoted by Ω_M, w.r.t. redshift z. The fractional density of radiation Ω_R is negligible in this era of the Universe.

where M is a constant having the dimension of mass and λ is a constant dimensionless parameter, which is set according to stability conditions.
The set of equations governing the cosmological dynamics of this k-essence model can be conveniently written in terms of three dimensionless parameters: [179, 182] :

$$x_1 = \frac{\kappa\dot{\varphi}}{\sqrt{6}H}\,, x_2 = \frac{\varphi^2\, e^{\kappa\lambda\varphi}}{2M^4}\,, x_3 = \frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H}\,, \tag{4.3}$$

where H is the Hubble-parameter and ρ_r is the density of radiation in the Universe. With these dimentionless parameters x_1, x_2 and x_3, the evolution equations can be cast in the following autonomous form :

$$\begin{aligned}\frac{dx_1}{dN} &= -\frac{x_1}{2}\frac{6(2x_2-1)+3\sqrt{6}\lambda x_1 x_2}{(6x_2-1)} \\ &\quad +\frac{x_1}{2}(3-3x_1^2+3x_1^2x_2+x_3)\,,\end{aligned} \tag{4.4}$$

$$\frac{dx_2}{dN} = x_2\frac{3x_2(4-\sqrt{6}\lambda x_1)-\sqrt{6}(\sqrt{6}-\lambda x_1)}{1-6x_2}\,, \tag{4.5}$$

$$\frac{dx_3}{dN} = -\frac{3}{2}(-x_1^2+x_1^2x_2+\frac{x_3^2}{3}+1)\,, \tag{4.6}$$

where $N = ln(\tilde{a}) = -ln(1+z)$; while $\tilde{a}$ and z are respectively the scale-factor and the redshift. N is generally called *e-foldings*. The advantage of these autonomous equations and the dimensionless parameters is that, these are easier to solve numerically, and various important cosmological quantities can be given in terms of these dimensionless parameters x_1, x_2 and x_3 *viz.* [179, 182],

$$w_{eff} = -1 - \frac{2\dot{H}}{3H^2} = -x_1^2 + x_1^2 x_2 + \frac{1}{3}x_3^2\,, \tag{4.7}$$

$$w_\varphi = \frac{1-x_2}{1-3x_2}\,, \tag{4.8}$$

$$c_s^2 = \frac{2x_2-1}{6x_2-1}\,, \tag{4.9}$$

$$\Omega_\varphi = -x_1^2 + 3x_1^2 x_2\,, \tag{4.10}$$

$$\Omega_R = x_3^2\,, \tag{4.11}$$

$$\Omega_M = 1 + x_1^2 - 3x_1^2 x_2 - x_3^2\,, \tag{4.12}$$

where w_{eff} and w_φ are respectively the effective equation-of-state parameter and the equation-of-state parameter of the k-essence model.
c_s is the sound speed of the k-essence model. Ω_φ, Ω_M and Ω_R are respectively the fractional densities of the dark energy, non-relativistic matter and radiation in the Universe. In the Fig. 4.1, the evolutions of the fractional densities Ω_φ and Ω_M have been depicted w.r.t. redshift z, for a certain period in the late Universe, with the initial conditions taken as $x_1 = 6.0 \times 10^{11}\,, x_2 = 0.5 + (1.0 \times 10^9)$, and $x_3 = 0.999$ at redshift $z \approx 10^{6.218}$ and the value of $\lambda = 0.2$ [182].

One can obtain simple equations for the Hubble-parameter H and the time t in terms of the e-foldings N, using equation 4.7 :

$$\frac{1}{H^2}\frac{dH}{dt} = -\frac{3}{2}(-x_1^2 + x_1^2 x_2 + \frac{1}{3}x_3^2 + 1)\,. \tag{4.13}$$

As $\frac{dN}{dt} = H$, the L.H.S. of the equation 4.13 can be expressed as :

$$\frac{1}{H^2}\frac{dH}{dt} = \frac{1}{H^2}\frac{dH}{dN}\frac{dN}{dt} = \frac{1}{H}\frac{dH}{dN}\,.$$

Denoting $h = ln\,H$, we get :

$$\frac{dh}{dN} = \frac{d}{dN}(ln\,H) = \frac{1}{H}\frac{dH}{dN}\,. \tag{4.14}$$

Using the equation 4.14, we can write the equation 4.13 as:

$$\frac{dh}{dN} = -\frac{3}{2}(-x_1^2 + x_1^2 x_2 + \frac{1}{3}x_3^2 + 1)\,, \tag{4.15}$$

by solving which we can get h and consequently $H = e^h$.

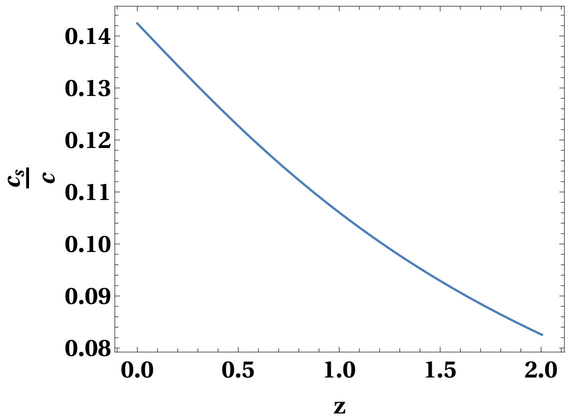

Figure 4.2: The variation of the sound speed c_s of the k-essence model w.r.t. redshift z.

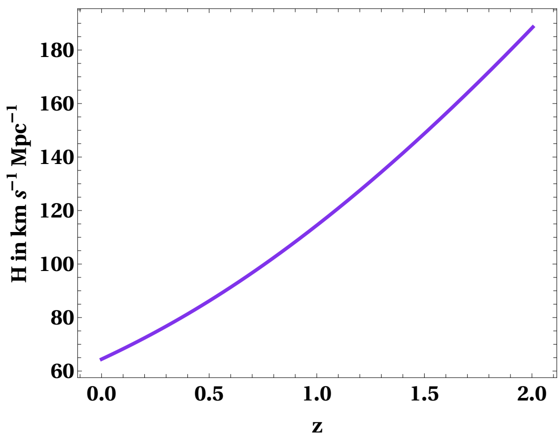

Figure 4.3: Evolution of the Hubble parameter H of the Universe, for the chosen k-essence dark energy model, w.r.t. redshift z.

After solving equation 4.15 for the Hubble-parameter, we can simply solve the equation:

$$\frac{dt}{dN} = \frac{1}{H}\,, \tag{4.16}$$

to get the time t. Also, for the scale factor $\tilde{a}$, we have the equation:

$$\frac{d\tilde{a}}{dN} = \tilde{a}\,. \tag{4.17}$$

Solving the above equation 4.17 one can obtain the scale factor $\tilde{a}$ and corresponding redshift from the relation $1 + z = 1/\tilde{a}$.

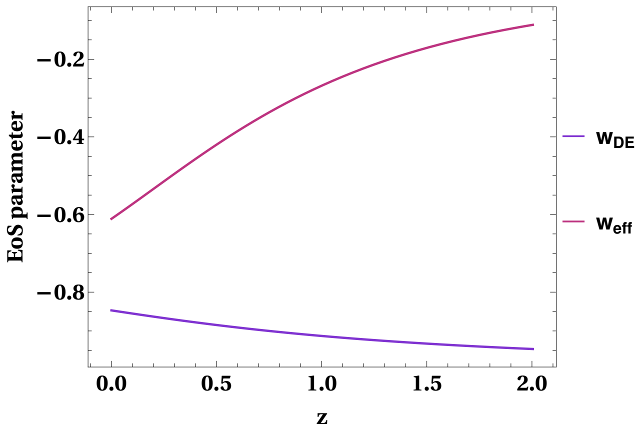

Figure 4.4: The variation of the equation-of-state (EoS) parameters $w_{DE} \equiv w_\varphi$ and w_{eff} w.r.t. redshift z.

We numerically solve the set of equations 4.4, 4.5, 4.6 along with the equations 4.15, 4.16 and 4.17, with appropriate initial conditions. We obtain the present values of the key cosmological parameters as : Hubble parameter $H_0 \approx 64.43\,km\,s^{-1}\,Mpc^{-1}$, fractional density of the k-essence dark energy $\Omega_{\varphi,0} \approx 0.721$, fractional density of non-relativistic matter $\Omega_{M,0} \approx 0.278$, equation-of-state parameter of the k-essence dark energy $w_{\varphi,0} \approx -0.847$, effective equation-of-state parameter $w_{eff,0} \approx -0.611$. The evolutions of the Hubble parameter H w.r.t. redshift and equation-of-state parameters w_φ, w_{eff} w.r.t. redshift have been plotted in the Fig.4.3 and Fig.4.4 respectively.
We now consider accretion of dark energy by black holes in the context of the above dark energy model. It may be noted here that the rate of accretion is affected by the sound speed of the ambient fluid. The surface of accretion is defined by the black hole horizon if there is no critical point outside the horizon [165]. The fluid being accreted by a black hole has the critical point, if its speed increases from subsonic to transonic values. From the historical development of spherical accretion by black holes starting from the pioneering work by Bondi [59], it is evident that if a black hole moves through the ambient medium with a speed much lesser than speed of light in vacuum (c), and the medium, considered as a perfect fluid, has a sound speed less than c, then the accretion radius would be larger than the black hole horizon i.e. the Schwarzschild radius [58]. In the above scenario where the sound speed of the k-essence model lies in the range $0 < c_s/c < 1$, the time-rate of change of mass of a black hole spherically accreting the k-essence dark energy is obtained by using the accretion radius $r_a = Gm/(v_{rel}^2 + c_s^2)$, which defines the relevant surface of accretion [58]. Hence, the rate of accretion is given by

$$\frac{dm}{dt} = 4\pi\mathscr{A}\frac{G^2m^2}{(v_{rel}^2 + c_s^2)^{3/2}}(1 + w_\varphi)\rho_\varphi\,, \tag{4.18}$$

where $c_s = \left(\frac{dP}{d\rho}\right)^{1/2}$ and v_{rel} is the relative speed of the black hole with respect to the ambient cosmic fluid being accreted. ρ_φ is the background density of the k-essence dark energy, w_φ is the equation-of-state parameter of the k-essence dark-energy. $\mathscr{A}$ is a proportionality-factor that can be taken to be of the order of ~ 1 [58]. Note that for $v_{rel} << c_s$ and $c_s \sim c$, the equation 4.18 leads to the rate of change of mass derived in Ref. [165].
For the present analysis we consider that the relative speed of the moving black hole with respect to the ambient cosmic fluid, *viz.*, the k-essence dark energy, is negligible in comparison to the

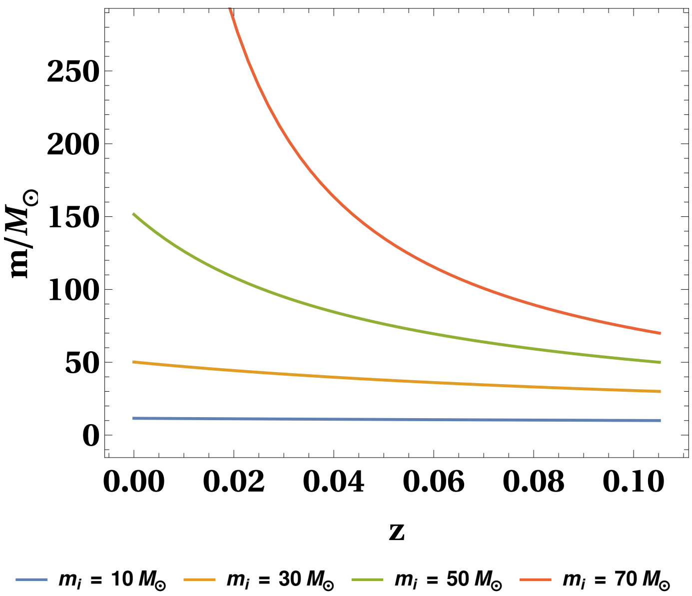

Figure 4.5: Growth of mass of black holes with various initial masses due to accretion of the k-essence dark energy w.r.t. redshift z.

sound speed of the k-essence model, *i.e.*, $v_{rel} << c_s$. This is valid for most of the black holes in late Universe, as it can be seen that in the dark energy dominated Universe, the sound speed of the chosen k-essence model is of the order of $\sim 0.1c$ (see Fig.4.2). So, in the denominator on the R.H.S. of equation 4.18, v_{rel}^2 can be neglected in comparison to c_s^2. Thereby, the time-rate of change of mass of a black hole due to spherical accretion of the k-essence dark energy is given by

$$\frac{dm}{dt} = 4\pi\mathscr{A}\frac{G^2m^2}{c_s^3}(1+w_\varphi)\rho_\varphi\,. \tag{4.19}$$

It may be noted from the evolution of the w_φ (Fig.4.4) that it decreases for higher redshift. Hence at higher redshift the w_φ makes the rate of growth of mass of a black hole lower.

We can determine the time-rate of change of mass of a black hole due to spherical accretion of the chosen k-essence dark energy using equation 4.19, where $\rho_\varphi = \rho_T\Omega_\varphi$, and $\rho_T = (3/8\pi G)H^2$ is obtained by solving the equation 4.15 for getting the Hubble-parameter H. We depict the evolution of masses of black holes, due to accretion of the chosen k-essence dark energy, with four different initial masses taken as $10, 30, 50$ and 70 times of the Solar-mass ($M_\odot$) respectively, with respect to the redshift z in Fig.4.5. It can be seen from the Fig.4.5 that the amount of growth in masses of the black holes due to the dark energy accretion increases with the increase in their initial masses. It may be noted that ordinary stellar-mass black holes, which are usually observed by electromagnetic signals emitted by various type of astrophysical mechanisms, generally have masses in the range $5-20\,M_\odot$. However, aLIGO and VIRGO have detected gravitational waves from mergers of binaries with component black holes having masses from 30 $M_\odot$ to as large as 80 $M_\odot$ [184, 185]. It is quite evident from the Fig.4.5 that stellar-mass black holes, having mass in the range $5-20\,M_\odot$, can grow to heavier ones by means of continuous spherical accretion of similar type of dark energy.

4.3 Power of gravitational waves emitted from binaries

The instantaneous power of gravitational radiation due to the orbital motion of two black holes of masses m_1 and m_2 in the quadrupole-approximation is given by [186, 12] :

$$\mathcal{P}(t) = \frac{8}{15}\frac{G^4}{c^5}\frac{M(m_1 m_2)^2}{(r_{min}(1+e))^5}(1+e\,Cos\phi)^4\left\{e^2 Sin^2\phi + 12(1+e\,Cos\phi)^2\right\}\,, \tag{4.20}$$

where $M = m_1 + m_2$, 'e' is the eccentricity of the orbit, ϕ is the angular position of the reduced mass μ on the plane of the orbit in a polar-coordinate system (r, ϕ) with origin at the center-of-mass, and r_{min} is the radial distance of closest approach. In the present case, the masses are continuously changing due to accretion of dark energy. Due to this continuous time-variation of the masses of the black holes, two extra terms (having single and double time-derivatives of the masses) arise along with the main term in the amplitude of gravitational radiation [187]. However, these terms are negligible in comparison to the main term in this case. Hence, the equation 4.20 needs to be considered here with time-dependent masses.

When the orbit is bounded, the total energy carried away by the gravitational radiation due to the relative motion of the system of two black holes within one complete cycle or time-period, is given by

$$\Delta\mathcal{E} = \int_0^T \mathcal{P}(t)dt = \int_0^{2\pi} \mathcal{P}(t)\frac{dt}{d\phi}d\phi\,. \tag{4.21}$$

It is known that energy of gravitational waves is well-defined when the average of the energy over several time-periods of the wave is taken. Also, a compact object in a Keplarian elliptical orbit emits gravitational waves with angular-frequencies, which are integral multiples of the angular-frequency $\omega_0 = (GM/a^3)^{1/2}$, where a is the semi-major axis of the elliptical orbit. Hence, the period of the gravitational waves emitted due to this orbital-motion, is a fraction of the orbital-period. Therefore, a well-defined version of the power of the emitted gravitational waves is the average of the power taken over one period of the orbit.

The average of the power P_{avg} over one period of the orbit can be written as [186]

$$P_{avg}(t) = \frac{1}{T}\int_0^T \mathcal{P}(t)dt = \frac{32G^4(m_1 m_2)^2 M}{5c^5 a^5} f(e)\,, \tag{4.22}$$

where the function $f(e)$ of eccentricity e is given by :

$$f(e) = \frac{1}{(1-e^2)^{7/2}}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right)\,. \tag{4.23}$$

For a circular orbit $e = 0$, thereby $f(e)$ becomes 1 and a becomes the radius of the circular orbit. Note that in case of constant masses the eccentricity of the orbit changes only due to the emission of gravitational waves. However, in the present case since the masses of the black holes are continuously changing through accretion of dark energy, the change of eccentricity would be due to two different effects: (i) growth of the masses via accretion, and (ii) loss of energy and angular momentum carried away by gravitational waves. The angular momentum of the system of two black holes is not affected due to spherical accretion of dark energy because the scalar-field dark energy model considered here does not contain angular momentum, and hence, can not impart any angular momentum to the system.

Using the rate of change of energy and angular momentum of a binary of black holes in bounded orbit, the rate of change of the semi-major axis a and eccentricity e of the orbit can be obtained as [20],

$$\frac{da}{dt} = -\frac{64}{5}\frac{G^3(m_1m_2)M}{c^5a^3}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) , \tag{4.24}$$

and

$$\frac{de}{dt} = -\frac{304}{15}\frac{G^3(m_1m_2)M}{c^5a^4}\frac{e}{(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right) . \tag{4.25}$$

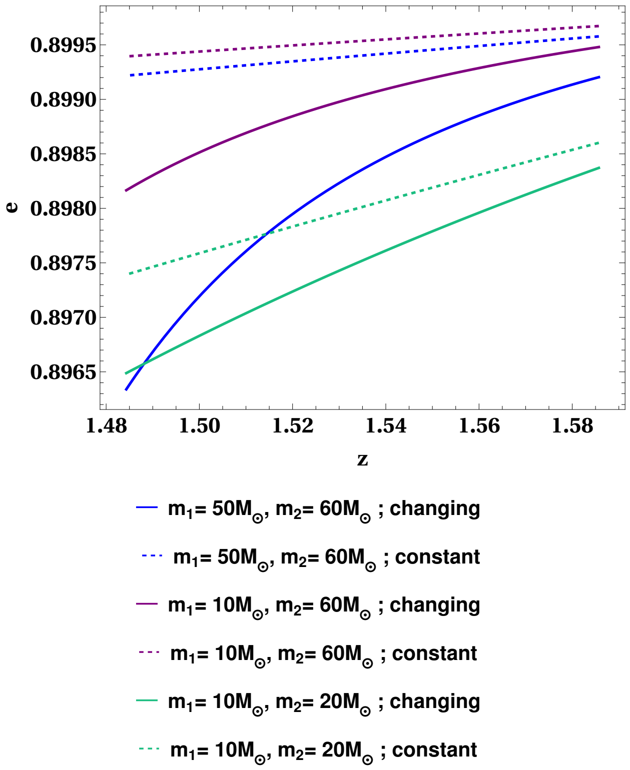

Figure 4.6: Evolution of eccentricities of elliptical orbits, from the initial value 0.9, w.r.t. redshift z, for three different combinations of the initial masses of black holes, for two different cases, (i) growing masses and (ii) constant masses.

It may be noted that the semi-major axis a and eccentricity e, governed by the above equations 4.24 and 4.25, are averages of these quantities over one period of the orbit, not their instantaneous values, as the corresponding equations of energy and angular momentum of the system, from which these are derived, govern their averages over one period. This is quite evident from the fact that these equations 4.24 and 4.25 do not contain the phase-angle ϕ.

From the equation 4.25 it follows that, if the eccentricity e becomes zero(0), then $\frac{de}{dt} = 0$ implying e = constant, *i.e.*, e remains zero. Thereby, once the orbit becomes circular, it remains circular. We solve these equations 4.24 and 4.25 numerically for different initial masses of the black holes forming binaries and orbiting in elliptical orbits with initial eccentricity $e_i = 0.9$. We choose the combinations of initial masses of the black holes forming the binaries to be $50, 60M_\odot\,; 10, 60M_\odot$ and $10, 20M_\odot$, respectively. The initial semi-major axis a_i of the elliptical orbit has been taken as 10^6 times of the sum of their initial Schwarzschild-radii, *i.e.*, $a_i = 10^6(2GM_i/c^2)$, (M_i being the initial total-mass of the black holes) so that the Keplarian-approximation holds well.

The time-period of the orbit is given by : $T = 2\pi/\omega_0$, where the angular-frequency ω_0 is given by : $\omega_0 = \left(\frac{GM}{a^3}\right)^{1/2}$.

The fall of the eccentricities of the elliptical orbits of the binaries, for three different combinations of initial masses of the constituent black holes, is depicted in the Fig.4.6 (Only certain portions of the full evolution-profiles have been shown here so that the differences can be visualized clearly). It can be seen from Fig.4.6 that the eccentricities for the orbits of binaries, where the masses of the black holes are growing due to accretion of the chosen model of k-essence dark energy, drop faster than those where the masses are constant. Moreover, the eccentricities of binaries with larger mass black holes drop faster.

After the eccentricity vanishes, *i.e.*, circularization of the orbit is achieved, the rate of change of

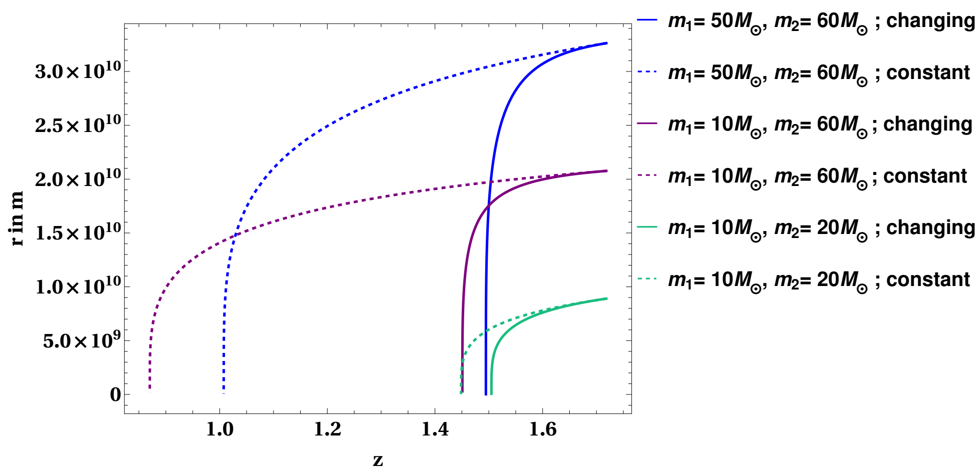

Figure 4.7: Variation of the radius r of circular orbit of two black holes in binary formation w.r.t. redshift z, in three different combinations of initial masses and two cases *viz.*, (i) growing masses, and (ii) constant masses.

radius r of the circular orbit is given by

$$\frac{dr}{dt} = -\frac{64}{5}\frac{G^3(m_1 m_2)M}{c^5 r^3}\,. \tag{4.26}$$

Correspondingly, the average power of the emitted gravitational wave for the circular orbit becomes,

$$P_{avg}(t) = \frac{32G^4(m_1 m_2)^2 M}{5c^5 r^5}\,. \tag{4.27}$$

We first determine the patterns of shrinking of radius r, by solving the equation 4.26, for the circular orbits in which two black holes of masses m_1 and m_2 are in binary formations, for two different cases *viz.*, (i) when the masses are changing due to spherical accretion of dark energy described by our chosen model and (ii) when the masses are constant, for three specified combinations of initial values of m_1 and m_2 for each of the cases. For this, we fix the initial radius for each of the cases to be 10^5 times of the sum of the initial Schwarzschild radii of the black holes, *i.e.*, $r_i = 10^5(2GM_i/c^2)$ (where M_i stands for the initial total mass of the black holes). This choice for the initial radii of the circular orbits for each case is considered to study the comparative evolution with similar initial conditions.

The radii of the circular orbits for three different combinations of initial values of masses m_1 and m_2, and for two different cases, as mentioned above, are plotted w.r.t. redshift z in Fig.4.7. It can be seen from the Fig.4.7 that, with the increasing difference in the masses and increasing total masses of the component black holes of the binaries, the difference in rate of shrinking of the radii of the circular orbits increases.

We next study the variation of the average power $P_{avg}(t)$ with the evolution of the circular orbits for each of the cases. We depict the variation of the average power w.r.t. the red-shift for each of the cases in Fig.4.8. It can be seen from the plots in Fig.4.8 that within the same interval, the average power of the emitted gravitational wave achieve significantly higher values for the binaries of black holes with growing masses, in comparison with the case when the masses of the black holes are constant.

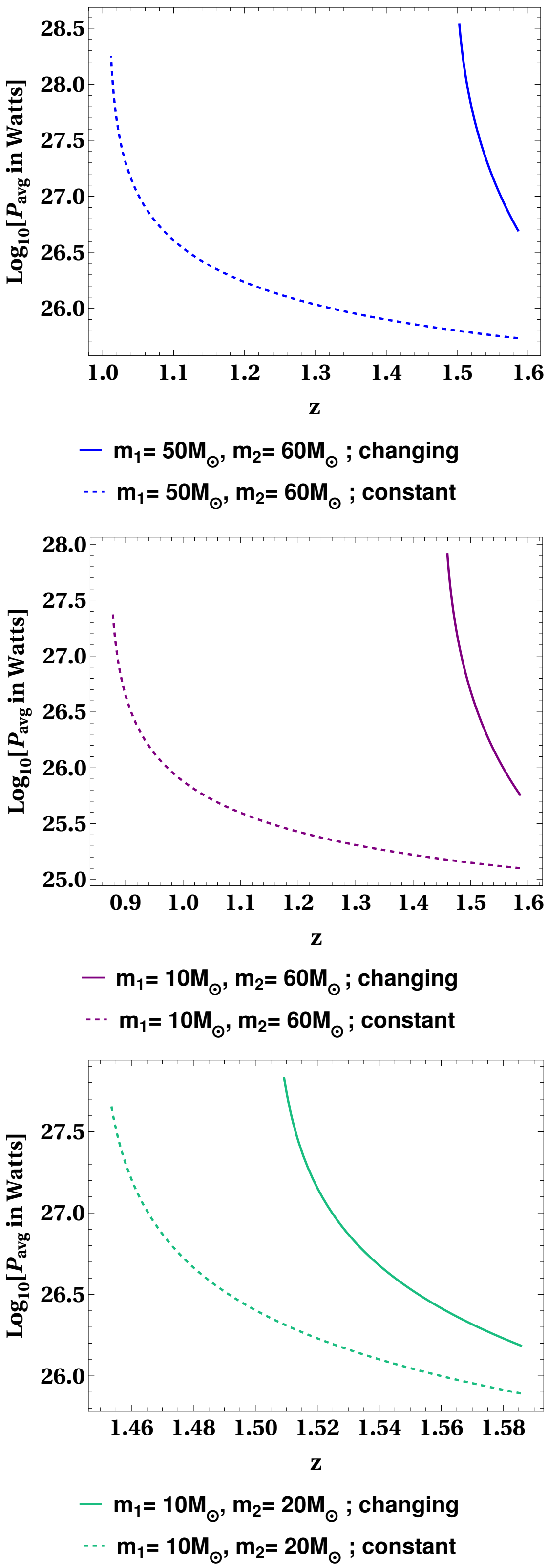

Figure 4.8: Evolution of average power P_{avg} of the gravitational wave emitted due to the orbital-motion of two black holes in binary formation, w.r.t. redshift z, for three different combinations of initial masses and for two different (constant and changing mass) cases. A range of the full evolution profiles have been shown for visual clarity.

The average power of the emitted gravitational wave in case of evolving masses of black holes in the binaries, grows faster in comparison to the case of constant masses of the black holes. A certain amount of increase of the masses of the black holes of binaries results in more amplification of the average power, because of the fact that the average power of the emitted gravitational waves is proportional to the quantity $\mu^2 M^3$ (μ being the reduced-mass of the black holes forming the binary). So, a small increment in the masses of the black holes results in a comparatively greater increase in the average power of the emitted gravitational waves. Moreover, the faster shrinking of the radius of circular orbit for increasing masses of the black holes, in comparison to the case of constant masses, also contributes to the faster growth of the average power $P_{avg}(t)$ in the former case, as it is proportional to r^{-5}.

4.4 Reduced coalescence time

From the previous analysis we have seen that as the masses of the black holes forming the binary increases due to accretion of dark energy, the average power of the emitted gravitational waves becomes significantly higher with the evolution of the orbit, in comparison to the case of constancy of masses when there is no accretion. Since the power of the emitted gravitational wave increases with time, the binary loses energy faster and shrinks more rapidly. As a result, the time taken by a binary to coalesce is shorter when the black holes' masses are growing, than for the case of constant masses of the black holes. Let us now estimate the decrease in coalescence time-interval of a binary, due to increasing masses of the component black holes that are spherically accreting the chosen model of k-essence dark energy.

For a binary constituted with black holes of constant masses, the rate of loss of energy by the binary is equal to the power of the emitted gravitational waves, *i.e.*, $P_{avg} = -dE_{avg}/dt$, where E_{avg} is the average energy of the binary. Using the expression of average power from equation 4.27, the time-evolution of the frequency of gravitational waves (f_{gw}) emitted from the binary is given by [20],

$$\frac{df_{gw}}{dt} = \frac{96}{5}\pi^{8/3}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3} f_{gw}^{11/3}\,, \tag{4.28}$$

where $\mathcal{M} = (m_1 m_2)^{3/5}/M^{1/5}$ is the chirp-mass of the binary. For the case of constant masses of the component black holes of the binary, the solution of the above equation 4.28 can be written as [20]:

$$f_{gw} = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8}\left(\frac{G\mathcal{M}}{c^3}\right)^{-5/8}\,, \tag{4.29}$$

where t_c is the time of coalescence of the binary and $\tau = t_c - t$ is the time-interval required by the binary to reach coalescence, from any stage of its evolution at an arbitrary time t. Using the equation 4.29 evaluated at an initial time t_i and the relation $\omega_{s\,i}^2 = (GM/r_i^3)$, where $\omega_{s\,i}$ is the initial angular-frequency of the source, it can be shown that the time-interval $\tau_i = t_c - t_i$ required by the binary to reach the coalescence stage from initial instant, is related to the initial radius of the circular-orbit r_i, as [20]

$$\tau_i = \frac{5}{256}\frac{c^5 r_i^4}{G^3 M m_1 m_2}\,. \tag{4.30}$$

Now, for the case when the masses of the black holes are changing, the counterpart of the equation 4.29 valid in the present case is given by

$$f_{gw}^{-8/3} = \frac{256}{5}\pi^{8/3}\left(\frac{G}{c^3}\right)^{5/3}\int_t^{t_c}\mathcal{M}^{5/3}dt\,. \tag{4.31}$$

Making a change of variable from t to τ in the integral $\int_t^{t_c}\mathcal{M}^{5/3}dt$, we can write:

$$\int_t^{t_c}\mathcal{M}^{5/3}dt = \int_0^{\tau}\mathcal{M}^{5/3}d\tau\,. \tag{4.32}$$

In following the suffix 'i' denotes the corresponding initial value of the quantity at initial time t_i'. Next, as was done for the case of constant masses, here also we evaluate the equation 4.31 at initial time t_i' and use the relation $\omega_{s\,i}^2 = (GM_i/r_i^3)$. It follows that t_i' or alternatively τ_i' satisfies the equation :

$$\frac{\int_0^{\tau_i'} \mathcal{M}^{5/3} d\tau}{\mathcal{M}_i^{5/3}} = \frac{5}{256} \frac{c^5 r_i^4}{G^3 M_i m_{1i} m_{2i}} , \tag{4.33}$$

as the counterpart of the equation 4.30 for the case of changing masses of black holes in binaries, due to dark energy accretion. The equations 4.30 and 4.33 provide the values of the coalescence time-intervals for the two different cases, *viz.*, constant mass and varying mass of the black holes, respectively. Therefore, when the initial masses of the component black holes and initial radii of orbits are same for both the cases, the difference of the coalescence time-intervals corresponding to the two cases is given by,

$$\Delta\tau_i = \tau_i - \tau_i'. \tag{4.34}$$

In order to perform a comparative estimate of the reduction in coalescence time-intervals due to dark energy accretion for the examples of binaries studied in the present work, we fix the initial time to be same for both the cases (pertaining to constant and changing masses), and evaluate the corresponding times of coalescence. We obtain the times of coalescence for binaries of the three different combinations of initial black hole masses considered by us in the previous section, for studying the evolution of eccentricities of the elliptical orbits, shrinking of radii of the circular orbits and the average power of emitted gravitational waves.
We choose the initial time at the e-folding value $N = -1$, or the corresponding redshift $z \approx 1.72$. For each of the cases, we set the initial radii r_i of the circular orbits to be 10^5 times of the sum of the initial Schwarzschild radii of the black holes. We display the decrease in coalescence time-intervals for these three different examples in the Table 4.1.

Initial masses	10 and 20 $M_\odot$	10 and 60 $M_\odot$	50 and 60 $M_\odot$
Initial radius of orbit r_i	$8.899 \times 10^9\, m$	$20.764 \times 10^9\, m$	$32.628 \times 10^9\, m$
t_c for constant masses	$4.817\ Gy$	$6.956\ Gy$	$6.329\ Gy$
t_c' for varying masses	$4.665\ Gy$	$4.81\ Gy$	$4.693\ Gy$
τ_i for constant masses	$66.135 \times 10^7\, y$	$2.8\, Gy$	$2.173\, Gy$
τ_i' for varying masses	$50.99 \times 10^7\, y$	$0.655\, Gy$	$0.537\, Gy$
$\Delta\tau_i = \tau_i - \tau_i'$	$15.14 \times 10^7\, y$	$2.146\, Gy$	$1.637\, Gy$

Table 4.1: Reduction in coalescence time-intervals due to accretion of the chosen model of dark energy.

From the above analysis it is evident that for binaries consisting of black holes, having masses in the stellar-mass range and few-times greater than the stellar-mass black holes, specifically those from which many merging events have been detected by the aLIGO and VIRGO detectors, the time required for coalescence gets significantly reduced due to the increase in masses of the black holes caused by accretion of the chosen model of dark energy. The magnitude of reduction in the coalescence time-interval is $\sim 10^8$ years, when both the component black holes are stellar-mass black holes (note the column for the combination of 10 and 20 $M_\odot$ in Table 4.1). However, for larger mass black holes (having masses few times larger than stellar-mass ones) the effect of accretion of the dark energy is greater. For example, the coalescence time-interval gets reduced by $\sim 10^9$ years (see the column for 50 and 60 $M_\odot$ in Table 4.1), and even more if there is a significant difference in the initial masses of the constituent black holes (see the column for 10 and 60 $M_\odot$ in Table 4.1). Note though, that the magnitude of decrease in coalescence time-interval also depends on the initial radius of the circular orbit.

It may be pertinent to mention that the black holes which are produced from stellar collapse may reside in regions of galaxies where they may also accrete other stellar matter, interstellar gas, dust etc.. During the accretion of such matter, characteristic electromagnetic radiation would be emitted. In the present era of *'multi-messenger astronomy'*, it should be possible to distinguish the effects of purely dark energy accretion on the gravitational wave spectrum, from the effects of stellar matter accretion, through these 'electromagnetic counterparts' of gravitational wave signals. Dark energy and dark matters, while getting accreted by black holes, should not emit any electromagnetic radiation.

Besides, the existence of accretion disks around black holes, accreting stellar matters, leads to certain characteristic phenomena, which would be absent in case of spherical accretion of dark energy or dark matter.

In the context of our present work, it is more significant to note the time-scale of the evolution of the binary from its formation to the merger. For black holes in stellar mass and few times higher than stellar mass ranges, and for sufficiently large initial separation, the coalescence time-interval is $\sim 10^8$ to 10^9 years. Continuous supply of stellar matters in the same order of density for accretion by the black holes seems far less likely throughout this time-scale. This is more obvious if the black holes are situated in the outer part of the galactic halo, which usually has very less density of stellar mass.

4.5 Conclusion and Discussion

A variety of cosmological observations have revealed that the present Universe is undergoing a phase of accelerated expansion, and such observations lend support to dynamical dark energy models responsible for the present acceleration. The string theory inspired dilatonic scalar field model, chosen in this work, in which acceleration is driven by the scalar field kinetic energy [179], seems to be observationally consistent [180]. If dark energy exists in an accretable form, it is inevitable that the black holes existing in the present Universe would evolve by accreting it. This in turn, would have a natural imprint on the evolution of binaries constituted by the black holes, as we have shown in this chapter.

Specifically, we have studied the effect of growth of masses of black holes due to the spherical accretion of the chosen model of k-essence dark energy on several important parameters of binaries constituted by those black holes. We have investigated the effect of changing masses of the black holes on the evolution of the binaries and the average power of the emitted gravitational waves. We have found that accretion of the chosen model of k-essence dark energy leads to rapid circularization of binary orbits in comparison to the case of constancy of masses i.e. without accretion. Further, in comparison with the constant-mass case, the average power of gravitational waves increases significantly faster due to the increase in masses of the black holes. Since the average power grows as $P_{avg} \propto \mu^2 M^3$, a comparatively small amount of increase in masses due to accretion leads to a much larger increment of the average power of emitted gravitational waves within the concerned time-scale. Finally, we have analysed how the effect of the increase in masses of the black holes leads to the reduction in the coalescence time-intervals of black hole binaries in the stellar mass range and above.

Our work establishes the fact that if dark energy is similar to scalar field models like the k-essence model considered here, then it would result in reduced coalescence time-intervals of the binaries of black holes present in the current era of the Universe. The reduction in coalescence time-intervals means increased rate of coalescences. However, the scenario of accretion of dark energy by supermassive black holes may be quite complicated. On the one hand, accretion of dark energy may itself be responsible for formation of supermassive black holes from primordial seeds leading to the the present large mass contained in supermassive black holes [188]. On the other hand, continued accretion of dark energy at a very high rate during the present epoch could result in further rapid rise in the mass of supermassive black holes leading to other astrophysical and cosmological issues such as depletion in the local density of dark energy surrounding such black holes. This would consequently indeed inhibit further accretion at some stage. Moreover, the supermassive black holes, usually residing at the centers of the galaxies, have various stellar matters surrounding it in its vicinity and hence those get accreted by it. Various astrophysical phenomena take place around the supermassive black holes due to accretion of these ambient stellar matters e.g. formation of accretion disks, relativistic jets, photon spheres etc.. These astrophysical processes around a supermassive black hole may prohibit scalar field dark energy to get spherically accreted by the supermassive black hole. So, the topic of dark energy acccretion by supermassive black holes is rather complex, and calls for further in-depth investigations.

A possible upshot of the effect of accretion of dark energy by black holes in binary formations is that if this effect is observationally detectable in the new era of gravitational wave astronomy, it can lead to independent constraints on the equation-of-state parameter w of the dark energy model. Such observations on local candidates are associated with much less noise compared to certain other dark energy observations involving all-sky surveys such as cosmic microwave background and baryon acoustic oscillations. Observations with aLIGO and VIRGO detectors should be useful in this regard, as we have demonstrated the significance of the effect for the binaries of black holes within mass-ranges, from which a considerable number of merging events have been detected by these detectors. Moreover, upcoming observations using the planned futuristic detector LISA, may also be able to investigate the imprint of dark energy accretion on coalescence time-intervals for binaries of supermassive black holes formed during galaxy mergers or even extreme mass-ratio inspirals (EMRIs) also.

Chapter 5

Evolution of axial perturbations in a non-rotating uncharged primordial black hole

5.1 Introduction

Perturbations in black hole space-times has been an interesting topic of research for the last few decades. At present, after the first direct detection of gravitational waves [5] and subsequent series of detections of gravitational waves from various similar sources [6, 7, 8, 9, 10], it is even more important for observational estimation of different parameters of gravitational wave sources, related to black holes, specially newly born black holes in 'ring-down phase' after merging of a binary of black holes. Black hole perturbation theory is one of the most important tools for accurately determining characteristics of this type of gravitational wave sources, through gravitational wave astronomy. The perturbations in a black hole's space-time can be created by various means e.g. (i) perturbations can be generated in the space-time of a black hole due to inspiralling motion of a comparatively very smaller particle (i.e. test-particle) around it ; (ii) the merging of two black holes in a binary creates perturbations in the newly born resultant black hole ; (iii) infall or interaction of gravitational waves from other sources can also create perturbations in a black hole's space-time etc.. In each of these different cases, black hole perturbation theory is pivotal to study the evolution of perturbations in the space-time of the black hole.

In 1957, Tullio Regge and John A. Wheeler derived the equation describing the behaviour or evolution of axial-perturbations in Schwarzschild metric [195] and using this, they studied the stability of the Schwarzschild geometry, when it is subjected to small non-spherical perturbations. This equation is known as 'Regge-Wheeler' equation after their name. This can be called effectively the birth of 'black hole perturbation theory'. Later S. Chandrasekhar derived the same equation in a different procedure and in a more general way[191].

Later in 1970, Zerilli extended the analysis to polar-perturbations in the Schwarzschild space-time [196, 197]. He showed that the equations governing the perturbations can be expressed as a pair of Schrödinger-like equations and he applied the formalism to study the gravitational radiation emitted by infalling test-particles into black holes.

The Regge-Wheeler and the Zerilli equations or perturbation techniques developed by them, and similar counterparts for other different types of black holes, paved the way to investigate the stability of perturbations for certain modes of vibration of the black holes. Instability or stability of perturbations mean whether the perturbations grow with time, thereby becoming too large to be handled by the linear perturbation theory or those decay gradually with time respectively. C. V. Vishveshwara analyzed the stability of the Schwarzschild black hole, through a numerical experiment, by slightly perturbing it with an infalling wave packet, thereby observing the scattered wave [198]. He found that the scattered wave is a sum of damped sinusoids, whose frequencies and damping times are the *'quasinormal modes'* i.e. the charateristic modes of free vibration of the black hole. The damping implies that the concerned black hole is stable i.e. returns into a stationary state after being perturbed. The outcome can be different for different types of black hole metrics and even can be different for same black hole metric with different environments, i.e. presence of some other matter near it.

In the present chapter, we derive the equation governing axial-perturbations in the space-time of a cosmological black hole, called 'generalized McVittie metric', proposed by V. Faraoni and A. Jacques in 2007 [190]. This is a generalization over the original 'McVittie metric', given by G. C. McVittie [189]. This generalized McVittie metric describes the space-time geometry of a Schwarzschild black hole embedded in FLRW-Universe, while allows change of mass of the black hole.

The reason for the choice of this metric in our work in this chapter, for describing the space-time around non-rotating uncharged primordial black holes (PBHs) has been described in the section 5.2. PBHs are thought to be produced in early Universe by direct gravitational collapse of regions with sufficient over-density after the horizon re-entry. Our main motivation is to study the evolution of perturbations in the space-time of a PBH in early radiation-dominated Universe, which has two distinct differences in comparison with the space-time of any astrophysical black hole in the present-era or late-time Universe. These differences are : (i) most of the PBHs in early Universe were subjected to rapid rate of change of mass due to either spherical accretion of the surrounding high-density radiation [199, 200, 201, 202] or due to Hawking evaporation and (ii) the effect of expansion of the Universe on the space-time around a PBH was significant due to the robust value of Hubble-parameter at that early era, in comparison to the late Universe. Due to these two differences, it is expected that the evolution of metric-perturbations around a PBH in early radiation-dominated Universe would be different than that of around an astrophysical black hole in the Universe of present era and that can not be explained by the usual equations viz. 'Regge-Wheeler equations' and 'Zerilli equations' for Schwarzschild metric or 'Tuekolsky master equations' for Kerr metric.

Propagation of gravitational waves in an expanding background in presence of a point-mass i.e. in Newtonian McVittie background has been investigated recently [203] and it has been shown that the point-mass increases the amplitude of the gravitational wave, while decreases its frequency, relative to an observer placed at infinity. However, a point-mass can hardly describe the more realistic scenario of the space-time around a PBH. Hence, exploration of the more generalized and relevant case of the generalized McVittie metric is necessary.

In deriving the desired equation, which governs the axial perturbations in the generalized McVittie metric, we have followed the basic procedure similar to that used by S. Chadrasekhar for deriving the Regge-Wheeler equation for the Schwarzschild metric. Yet, there are some fundamental differences. One of the main differences is that the unperturbed parameters, appearing in metric-coefficients of the generalized McVittie metric, are time-dependent. This is why the usual way of Fourier-transforming only the perturbations from time-space to frequency-space may not work here and this would ultimately give the intended equation as spatio-temporal equation, which would be although mathematically correct, but physically very hard to solve and interpret. For this reason, we employ Fourier-transformation of the overall perturbation equations. Then, applying various results of the 'Convolution theorem' for Fourier-transform of product of two functions and using some approximations, we get the equation in desired form. After transforming the equation in Schrödinger-like form, we identify the potential from this form of the equation and then analyze some important aspects of it.

The chapter is organized as follows : in the section 5.2, we describe the reason for our choice of the generalized McVittie metric for describing the space-time around a non-rotating, uncharged PBH, which is subjected to change of mass in the early radiation-dominated Universe. In this section we also describe the convenient forms of the line-element of this metric in different coordinates. In the section 5.3, we describe the derivation of the equation governing the axial-perturbations in the generalized McVittie metric to a certain level, and in the section 5.4 we simplify that equation by applying some approximations. We also separate the radial and angular parts of the equation with the application of separation of variables technique, where the radial part is the intended equation, as the angular part is identical to that of the Schwarzschild metric. Furthermore, in this section we transform the equation in a form free of first-order derivative term i.e. the Schrödinger-like form, from which we identify the potential to draw some physical interpretation from it. In the last section 5.5, we give final remarks of our work in this chapter. Besides these, there are five Appendices to complement and clarify various aspects of this work.

It is to be noted that we have mainly followed the natural system of units in the analytical calculations, where 'G' and 'c' are set to 1 or omitted. But, during discussing some numerical-orders, we use G and c in their necessary places in the expressions.

5.2 The reason for choice of the generalized McVittie metric and its forms in different coordinate systems

To derive the equation governing the axial-perturbations in space-time around a PBH in early Universe, first of all the correct metric describing the space-time is to be choosen. As in the early Universe, when the PBHs were produced and the era in which we are interested, the rate of expansion of the Universe i.e. the Hubble-parameter was very high, in comparison with the present era. Therefore, this effect of robust cosmological expansion on the local space-time around the PBHs can not be neglected. There were series of efforts to describe this effect and hence to get a resultant metric, which can describe the space-time of a black hole embedded in an expanding Universe, when the effect of expansion of the Universe on the space-time around the black hole is significant. This type of metrics are usually referred as 'cosmological black hole metrics'.

One of the first attempts was by McVittie [189]. The solution, given by him, is known as 'McVittie metric'. But, this metric can describe the intended space-time, provided the mass of the black hole is not changing with time. However, in the case of PBHs in early Universe, this condition of constancy of mass would be impractical because, there was highly dense radiation almost everywhere in the radiation-dominated era and hence this high-density radiation was subject to spherical accretion by the PBHs, leading to the growth of masses of the PBHs. Also, as PBHs spanned an enormous mass-range, from the end of inflation ($10^{-32}\,s$) up to the big bang nucleosynthesis ($\sim 1\,s$) [204] and a large fraction of the PBHs, created in early Universe, were of smaller mass than the Solar-mass ; Hawking-evaporation was a prominent phenomenon for those.

Some PBHs were in the mass-range such that for them the rate of loss of mass due to Hawking

radiation should be very dominant, such that the rate of gain of mass due to spherical accretion of the surrounding radiation would be insignificant in comparison to the mass-loss rate due to Hawking evaporation. Again, there were some PBHs in the mass range such that for them the rate of loss of mass due to Hawking evaporation was negligible with respect to the rate of mass-gain by spherical accretion of high-density radiation. So, there is no doubt that most of the PBHs were in the mass range such that they were undergone mass change with time, whether was it gain of mass or loss of mass. So, the condition of constancy of mass is not justified at all for those PBHs. That is why there were more works following the work of McVittie, trying to give a more generalized solution where there is not any restriction on the change of mass. One of the metrics, which has been subject of much interest, is the 'Schwarschild-De Sitter metric'. When the background metric is chosen to be De Sitter, the McVittie metric reduces to Schwarzschild-De Sitter metric. The 'Schwarschild-Anti De Sitter metric' is also of similar interest. But, these are vacuum solutions of Einstein's equations and hence these do not allow the mass-change of PBHs due to accretion of surrounding radiation, in the early rdaiation-dominated era of the Universe.
Another two attempts, to describe such black hole metrics, are the Sultana-Dyer solution [205] and MaClure-Dyer solution [206]. But, each of these has their own shortcomings to describe the metric of a black hole embedded in an expanding FLRW-Universe. Their deficiencies have been briefly described in references [190, 207].
In 2007, V. Faraoni and A. Jacques [190] gave a new solution, better to say they proposed a more generalized extension of the McVittie metric. They showed that it can describe mass-change of a Schwarzschild black hole embedded in an expanding FLRW-Universe, due to spherically accreting surrounding cosmic-fluid (viz. radiation in this case). They called it the 'Generalized McVittie metric'. More explanation and emphasis on this metric was given in a subsequent work [192].
For the present purpose, we choose this new metric, as it does not require any imposed condition on the change of mass of the black holes, while also it does not seem to have any major theoretical drawbacks.
This metric is written in isotropic coordinates as:

$$ds^2 = -\frac{\mathcal{B}^2(t,r)}{\mathcal{A}^2(t,r)}dt^2 + a^2(t)\mathcal{A}^4(t,r)(dr^2 + r^2 d\Omega^2)\,, \tag{5.1}$$

where, r and t are respectively the isotropic radial and time coordinates. The quantities $\mathcal{A}$ and $\mathcal{B}$ are given by :

$$\mathcal{A}(t,r) = 1 + \frac{m(t)}{2r} \text{ and } \mathcal{B}(t,r) = 1 - \frac{m(t)}{2r},$$

where $m(t)$ is the mass of the PBH and it is time-varying. $d\Omega^2 = r^2(d\theta^2 + Sin^2\theta d\phi^2)$ is the usual angular part.
For convenience, the authors of reference [190] introduced the quantity :

$$\mathcal{C} = \left(\frac{\dot{a}}{a} + \frac{\dot{m}}{r\mathcal{A}}\right) = \frac{\dot{M}_H}{M_H} - \frac{\dot{m}}{m}\frac{\mathcal{B}}{\mathcal{A}}\,, \tag{5.2}$$

where M_H represents the 'Hawking-Hayward quasi-local mass' of the black hole and $a(t)$ is the scale-factor of the background FLRW-Universe. The 'dot' denotes differentiation with respect to the isotropic time coordinate t.
This metric 5.1 can be transformed to a Schwarzschild-like coordinate system. C. Gao et al have described this process of transforming the metric into a Schwarzschild-like form, in their work

[192]. Yet we are mentioning it briefly as the definitions and inter-relations of corresponding coordinates are required in our work. First by defining the areal radius

$$\tilde{r} = r\Big(1 + \frac{M_H(t)}{2ra(t)}\Big)^2 \tag{5.3}$$

and using $m(t) = M_H(t)/a(t)$, and then introducing the co-moving radial coordinate $R = a\tilde{r}$, in terms of which $d\tilde{r} = \dfrac{dR}{a} - H\tilde{r}dt$, equation 5.1 is turned into the Painleve-Gullstrand form :

$$\begin{aligned} ds^2 = -\left\{1 - \frac{2M_H}{R} - \frac{\Big(HR + \dot{m}a\sqrt{\frac{\tilde{r}}{r}}\Big)^2}{1 - \frac{2M_H}{R}}\right\} dt^2 + \\ \frac{1}{1-\frac{2M_H}{R}} dR^2 + R^2 d\Omega^2 - \frac{2}{1-\frac{2M_H}{R}}\left\{HR + \dot{m}a\sqrt{\frac{\tilde{r}}{r}}\right\} dtdR\,, \end{aligned} \tag{5.4}$$

where H is the Hubble parameter. Further setting

$$A(t,r) = 1 - \frac{2M_H}{R}\,, \tag{5.5}$$

$$C(t,r) = HR + \dot{m}a\sqrt{\frac{\tilde{r}}{r}}\,, \tag{5.6}$$

and defining the time coordinate $\bar{t}$ as

$$d\bar{t} = \frac{1}{F}\Big(dt + \frac{C}{A^2 - C^2}dR\Big)\,, \tag{5.7}$$

where $F(t,r)$ is an integrating-factor that makes $d\bar{t}$ an exact-differential , one gets

$$\begin{aligned} ds^2 = -\frac{(A^2 - C^2)}{A}\left\{F^2 d\bar{t}^2 + \frac{C^2 dR^2}{(A^2-C^2)^2} - \frac{2FC}{(A^2-C^2)} d\bar{t}dR\right\} \\ + \frac{dR^2}{A} + R^2 d\Omega^2 - \frac{2C}{A} dR\Big(F d\bar{t} - \frac{C}{(A^2-C^2)}dR\Big)\,. \end{aligned} \tag{5.8}$$

The cross terms containing $dRd\bar{t}$ cancel out and the squared line-element in the new 'Nolan-gauge' becomes

$$ds^2 = -A\Big(1 - \frac{C^2}{A^2}\Big)F^2 d\bar{t}^2 + A^{-1}\Big(1 - \frac{C^2}{A^2}\Big)^{-1} dR^2 + R^2 d\Omega^2\,. \tag{5.9}$$

We are using this form 5.9 in our work discussed in this chapter. There may be a question that why we choose to work with this form 5.9 in 'Nolan gauge', which is a Schwarzschild-like form, instead of using the form 5.1 in isotropic coordinate system, despite the fact that working in isotropic-coordinate system seems to be comparatively simpler. The main reason is that we want to utilize the symmetry of the form 5.9 viz. if we see the time coordinate as $\tilde{t}$ such that $d\tilde{t} = Fd\bar{t}$, then $g_{00} = -g_{11}^{-1}$, where the indices '0' and '1' stand for temporal and radial coordinate respectively. Furthermore, the isotropic radial coordinate r does not always faithfully represent radial distances. This fact is very disturbing while interpreting the results with practical cases.

5.3 Derivation of equations describing the axial perturbations in Generalized McVittie metric

The square of line-element of a generalized metric can be written as :

$$ds^2 = -e^{2\nu}d\tau^2 + e^{2\psi}(d\phi - \omega d\tau - q_2 dx_2 - q_3 dx_3)^2 + e^{2\mu_2}dx_2^2 + e^{2\mu_3}dx_3^2\,, \tag{5.10}$$

where the quantities $\nu, \psi, \mu_2, \mu_3, \omega, q_2, q_3$ are functions of the coordinates τ, x_2, x_3. Now we compare the form of the metric given in the equation 5.10 with the generalized McVittie metric proposed by V. Faraoni et al [190] in Nolan Gauge, as given in equation 5.9, with the corresponding coordinates being $\tau \equiv \bar{t}\,, x_2 \equiv R\,, x_3 \equiv \theta$ and $\phi \equiv \phi$ (We follow the index designation : $0, 1, 2, 3$ stand for respectively $\bar{t}, \phi, R, \theta$). Comparing the metric in equation 5.10 with the generalized McVittie metric in Nolan Gauge, as given in equation 5.9, we see that the coefficients $\nu, \psi, \mu_2, \mu_3, \omega, q_2, q_3$, in case of generalized McVittie metric are given by the set of equations :

$$-e^{2\nu} + e^{2\psi}\omega^2 = -\Big(1 - \frac{2M_H}{R}\Big)\Big(1 - \frac{C^2}{A^2}\Big)F^2\,, \tag{5.11}$$

$$e^{2\psi}q_2^2 + e^{2\mu_2} = \Big(1 - \frac{2M_H}{R}\Big)^{-1}\Big(1 - \frac{C^2}{A^2}\Big)^{-1}\,, \tag{5.12}$$

$$(e^{2\psi}q_3^2 + e^{2\mu_3}) = R^2\,, \tag{5.13}$$

$$e^{2\psi} = R^2 Sin^2\theta\,. \tag{5.14}$$

[1] While the absence of cross-terms (i.e. terms with $dx_i dx_j$, $d\tau dx_i, dx_i d\phi$ etc.) in the metric 5.9 indicate that the zeroth order or unperturbed values of the coefficients causing the cross terms are zero *viz.* comparing with the metric given in equation 5.10 : $\omega = 0\,, q_2 = 0\,, q_3 = 0$.
Then solving the above set of equations 5.11 to 5.14 for the coefficients ν, ψ, μ_2 and μ_3, we obtain :

$$\nu = \frac{1}{2}ln\left\{\Big(1 - \frac{2M_H}{R}\Big)\Big(1 - \frac{C^2}{A^2}\Big)F^2\right\}\,, \tag{5.15}$$

$$\mu_2 = \frac{1}{2}ln\left\{\Big(1 - \frac{2M_H}{R}\Big)^{-1}\Big(1 - \frac{C^2}{A^2}\Big)^{-1}\right\}\,, \tag{5.16}$$

$$\mu_3 = \frac{1}{2}ln(R^2)\,, \tag{5.17}$$

$$\psi = \frac{1}{2}ln(R^2 Sin^2\theta)\,. \tag{5.18}$$

Now, we have to get the expressions of the Ricci tensor components R_{12} and R_{13}, upto the first order perturbations of the quantities ω, q_2 and q_3. Here we denote the first order or linear perturbations in $\omega\,, q_2$ and q_3 as $\delta\omega\,, \delta q_2$ and δq_3 respectively. But, as we have already stated that in this case the background or zeroth order values of these quantities are zero : $\omega = q_2 = q_3 = 0$, hence the overall quantities can be given by $\delta\omega\,, \delta q_2$ and δq_3. The axial perturbations (as called in reference [191]) are characterized by the non-zero values of $\delta\omega\,, \delta q_2$ and δq_3. Any general perturbation of this metric 5.10 would generate the perturbations $\delta\omega\,, \delta q_2$ and δq_3, with zero

[1] One issue is to be noted here that the metric signatures of the metric in form 5.10 is opposite to that used in reference [191]. For this reason in each of the expressions we are using here, there will be a negative multiplicity with $e^{2\alpha}$ ($\alpha = \nu, \psi, \mu_2, \mu_3$). While except this sign change, there would not be any other change in the expressions of the Ricci tensors.

unperturbed values for the case without any cross-terms. While the non-zero unperturbed values of the quantities $\nu\,,\psi\,,\mu_2\,,\mu_3$ would experience first order perturbations $\delta\nu\,,\delta\psi\,,\delta\mu_2$ and $\delta\mu_3$ respectively. But, as in the case of Schwarzschild metric, in this case of generalized McVittie metric too, these two sets of perturbations have completely different effects. As argued in reference [191], the set of perturbations $\delta\omega\,,\delta q_2$ and δq_3 induce a dragging of the inertial frame thereby imparting a rotation on the black hole. But, the other set has no such rotational effects. For this reason they are respectively called as Axial and Polar perturbations in reference [191], on the basis of effect of sign-reversal of ϕ on the metric. On the basis of this different behaviour, we can physically interpret that they must decouple .

For this reason instead of getting the equations, which govern all the perturbations in a metric, in detail i.e. where both the sets of perturbations will be present, we can extract the part containing the set $\delta\omega\,,\delta q_2$ and δq_3 first and then the the part containing the other. The part containing the set of perturbations $\delta\omega\,,\delta q_2$ and δq_3, will contain the background values of quantities $\nu\,,\psi\,,\mu_2\,,\mu_3$. Following reference [191], we use the definitions :

$$Q_{AB} = \delta q_{A,B} - \delta q_{B,A}\,, \tag{5.19}$$

$$Q_{A0} = \delta q_{A,0} - \delta\omega_{,A}\,, \tag{5.20}$$

where the indices A, B = 2,3 in this case.

[2] Using the expressions given in the reference [191], the components of Ricci tensor $R_{12}+\delta_{\omega,q_2,q_3}R_{12}$ and $R_{13}+\delta_{\omega,q_2,q_3}R_{13}$ (these give the first order perturbations only, because the background values of these Ricci tensor components R_{12} and R_{13} are zero ;) in our case are given by :

$$R_{12}+\delta_{\omega,q_2,q_3}R_{12} = \frac{1}{2}e^{-2\psi}(e^{-2\nu}e^{-2\mu_3})^{1/2}[((e^{3\psi+\nu-(\mu_2+\mu_3)})Q_{32})_{,3} - ((e^{3\psi-\nu-\mu_2+\mu_3})Q_{02})_{,0}]\,, \tag{5.21}$$

and

$$R_{13}+\delta_{\omega,q_2,q_3}R_{13} = \frac{1}{2}e^{-2\psi}(e^{-2\nu}e^{-2\mu_2})^{1/2}[((e^{3\psi+\nu-(\mu_2+\mu_3)})Q_{23})_{,2} - ((e^{3\psi-\nu+\mu_2-\mu_3})Q_{03})_{,0}]\,. \tag{5.22}$$

Before proceeding we need the values of the perturbation to the metric components g_{12} and g_{13} i.e. $\delta_{\omega,q_2,q_3}g_{12}$ and $\delta_{\omega,q_2,q_3}g_{13}$. It is to be noted that the metric is given in covariant form in the line-element 5.10. So, the perturbation to the metric components w.r.t. $\omega\,,q_2$ and q_3 are given by : $\delta_{\omega,q_2,q_3}g_{12} = -e^{2\psi}\delta q_2$ and $\delta_{\omega,q_2,q_3}g_{13} = -e^{2\psi}\delta q_3$.

Now, we start getting the equations describing the axial perturbations for the generalized McVittie metric. The origin of the equations has been discussed in APPENDIX 2 5.7.

The equation $\delta_{\omega,q_2,q_3}R_{12} - \frac{1}{2}(\delta_{\omega,q_2,q_3}g_{12})\mathcal{R} = 0$ is :

$$-\frac{1}{2}e^{-2\psi}(e^{-2\nu}e^{-2\mu_3})^{1/2}[((e^{3\psi+\nu-(\mu_2+\mu_3)})Q_{23})_{,3} + ((e^{3\psi-\nu-\mu_2+\mu_3})Q_{02})_{,0}] = \frac{1}{2}(-e^{2\psi}\delta q_2)\mathcal{R}\,, \tag{5.23}$$

and the equation $\delta_{\omega,q_2,q_3}R_{13} - \frac{1}{2}(\delta_{\omega,q_2,q_3}g_{13})\mathcal{R} = 0$ is :

$$\frac{1}{2}e^{-2\psi}(e^{-2\nu}e^{-2\mu_2})^{1/2}((e^{3\psi+\nu-(\mu_2+\mu_3)})Q_{23})_{,2} - ((e^{3\psi-\nu+\mu_2-\mu_3})Q_{03})_{,0} = \frac{1}{2}(-e^{2\psi}\delta q_3)\mathcal{R}\,. \tag{5.24}$$

The $\mathcal{R}$ in the above equations is the Ricci-scalar. Although we shall see later, that $\mathcal{R}$ vanishes in the scenario of our interest, yet we keep the $\mathcal{R}$ in the equations to a certain stage, so that the

[2] *Our notation of representing the Ricci tensor components, upto first order perturbation of the quantities $\omega\,,q_2$ and q_3 is : $R_{ij}+\delta_{\omega,q_2,q_3}R_{ij}$.*

appearance of the $\mathcal{R}$ in the desired final equation can be checked. In any more general case, where the $\mathcal{R}$ does not vanish, this may be utilized.
After shifting the factor $e^{-2\psi}$ from LHS to RHS of the equations 5.23 and 5.24, we write these as respectively :

$$-\frac{1}{2}(e^{-2\nu}e^{-2\mu_3})^{1/2}[((e^{3\psi+\nu-(\mu_2+\mu_3)})Q_{23})_{,3}+((e^{3\psi-\nu-\mu_2+\mu_3})Q_{02})_{,0}]=\frac{1}{2}e^{4\psi}(-\delta q_2)\mathcal{R}\,, \tag{5.25}$$

and

$$\frac{1}{2}(e^{-2\nu}e^{-2\mu_2})^{1/2}((e^{3\psi+\nu-(\mu_2+\mu_3)})Q_{23})_{,2}-((e^{3\psi-\nu+\mu_2-\mu_3})Q_{03})_{,0}=\frac{1}{2}e^{4\psi}(-\delta q_3)\mathcal{R}\,. \tag{5.26}$$

Before proceeding we define some quantities for brevity and compactness of the upcoming equations, as was defined in reference [191] for Schwarzschild metric, as below :

$$Q(\bar{t},R,\theta)=Q_{23}\Delta Sin^3\theta\,, \tag{5.27}$$

where the quantity Δ is given by :

$$\Delta=R^2-2M_HR\,. \tag{5.28}$$

Hence,

$$\Big(1-\frac{2M_H}{R}\Big)=\Big(\frac{R^2-2M_HR}{R^2}\Big)\equiv\frac{\Delta}{R^2}\,. \tag{5.29}$$

Following expressions of the unperturbed coefficients present in the metric of form 5.10 for the metric 5.9 are useful :

$$e^{3\psi}=R^3\,Sin^3\theta\,,$$
$$e^{\nu}=\Big(1-\frac{2M_H}{R}\Big)^{1/2}\Big(1-\frac{C^2}{A^2}\Big)^{1/2}F\,,$$
$$e^{-\mu_3}=R^{-1}\,,$$
$$e^{\mu_2}=\Big(1-\frac{2M_H}{R}\Big)^{-1/2}\Big(1-\frac{C^2}{A^2}\Big)^{-1/2}\,.$$

Now, substituting the expressions of $e^{2\alpha}$ $(\alpha=\nu,\psi,\mu_2,\mu_3)$, in the equations 5.25 and 5.26, we obtain the quantities present in these equations for the metric given in equation 5.9. According to the convention and notation of S. Chandrasekhar in reference [191], the quantity Q_{02} is given by :

$$Q_{02}=-Q_{20}=(\delta\omega_{,2}-\delta q_{2,0})\,. \tag{5.30}$$

From now, we shall denote the quantity $F\Big(1-\frac{C^2}{A^2}\Big)$ as λ for brevity. Hence, the equation 5.25 in our case (rearranging it in a form of our convenience) can be written as :

$$\begin{aligned}\frac{F\lambda}{R^4\,Sin^3\theta}\frac{\partial Q}{\partial\theta}&=F^2R\,Sin\,\theta\,\Big(1-\frac{2M_H}{R}\Big)^{\frac{1}{2}}\Big(\frac{\lambda}{F}\Big)^{\frac{1}{2}}\mathcal{R}\delta q_2\\&-\Big(\frac{\partial^2\delta\omega}{\partial\bar{t}\partial R}-\frac{\partial^2\delta q_2}{\partial\bar{t}^2}\Big)-\Big(\frac{\partial\delta\omega}{\partial R}-\frac{\partial\delta q_2}{\partial\bar{t}}\Big)\frac{F}{R^4}\frac{\partial}{\partial\bar{t}}\Big(\frac{R^4}{F}\Big)\,.\end{aligned} \tag{5.31}$$

On the other hand, equation 5.26 in our case, for the metric given in equation 5.9, can be conveniently written as below :

$$\begin{aligned} \frac{\Delta}{R^4\,Sin^3\theta}\left(\frac{\partial Q}{\partial R}+\frac{Q}{\lambda}\frac{\partial\lambda}{\partial R}\right)-\frac{1}{\lambda^2}\frac{\partial}{\partial\bar{t}}\Big(\frac{\partial\delta\omega}{\partial\theta}-\frac{\partial\delta q_3}{\partial\bar{t}}\Big) \\ -\frac{\Delta}{R^4\lambda\,Sin^3\theta}\Big(\frac{\partial\delta\omega}{\partial\theta}-\frac{\partial\delta q_3}{\partial\bar{t}}\Big)\frac{\partial}{\partial\bar{t}}\Big(\frac{R^4\,Sin^3\theta}{\Delta\,F}\Big(\frac{\lambda}{F}\Big)^{-1}\Big)=-\delta q_3\Delta\,Sin\,\theta\left(\frac{\lambda}{F}\right)^{-1}\mathcal{R}\,. \end{aligned} \tag{5.32}$$

At this point, we shall face difficulty if we assume that the perturbations have time-dependence proportional to $e^{i\sigma\bar{t}}$, where σ is a constant, or, in other way to say, if we take the Fourier-transform of the perturbations from $\bar{t}$-space to σ-space. Then the resulting terms will not serve the purpose. This is because in the coordinate system in Nolan-gauge, which we are using, the time-coordinate $\bar{t}$ is a function of both the isotropic radial and time coordinates viz. r and t. Also, the radial coordinate R and the temporal coordinate $\bar{t}$ are inter-related. For the same reason, in the term $\frac{\partial^2\delta\omega}{\partial\bar{t}\partial R}$, the partial derivatives w.r.t. R and $\bar{t}$ can not be commuted.

So, to avoid this complication in the calculations, we shall convert the partial derivatives w.r.t. R and $\bar{t}$ into those w.r.t. r and t respectively.

Expressing the following double-partial derivatives of the perturbations in terms of partial derivatives w.r.t. r and t (using the formula given in APPENDIX-3 i.e. section 5.8), we obtain :

$$\frac{\partial^2\delta\omega}{\partial\bar{t}\partial R}=\left\{\lambda\frac{\partial}{\partial t}\Big(a^{-1}\Big(1-\frac{M_H^2}{4a^2r^2}\Big)^{-1}\Big)\right\}\frac{\partial\delta\omega}{\partial r}+\left\{\lambda\Big(a^{-1}\Big(1-\frac{M_H^2}{4a^2r^2}\Big)^{-1}\Big)\right\}\frac{\partial^2\delta\omega}{\partial t\partial r}\,, \tag{5.33}$$

and

$$\frac{\partial^2\delta q_2}{\partial\bar{t}^2}=\left(\lambda\frac{\partial\lambda}{\partial t}\right)\frac{\partial\delta q_2}{\partial t}+\lambda^2\frac{\partial^2\delta q_2}{\partial t^2}\,. \tag{5.34}$$

We use the above expressions of the partial derivatives of $\delta\omega$ and δq_2 from the equations 5.33 and 5.34 in the equation 5.31, and then we Fourier-transform both sides of the equation from time-space to frequency-space. For brevity, from now we designate the quantity $\frac{\Delta\lambda}{R^4}\frac{\partial}{\partial\bar{t}}\left(\frac{R^4}{\Delta\lambda}\right)=\mathbb{F}(r,t)$ and the quantity $\frac{F}{R^4}\frac{\partial}{\partial\bar{t}}\left(\frac{R^4}{F}\right)=\mathscr{F}(r,t)$. Thus we obtain :

$$\begin{aligned} \frac{1}{2\pi}\int_0^\infty\frac{F\lambda}{R^4\,Sin^3\theta}\frac{\partial Q}{\partial\theta}\,e^{-i\sigma t}dt=\frac{1}{2\pi}\int_0^\infty F^2R\,Sin\,\theta\Big(1-\frac{2M_H}{R}\Big)^{\frac{1}{2}}\Big(\frac{\lambda}{F}\Big)^{\frac{1}{2}}\mathcal{R}\delta q_2\,e^{-i\sigma t}dt \\ -\frac{1}{2\pi}\int_0^\infty\left\{a^{-1}\Big(1-\frac{M_H^2}{4a^2r^2}\Big)^{-1}\frac{\partial\delta\omega}{\partial r}-\lambda\frac{\partial\delta q_2}{\partial t}\right\}\mathscr{F}\,e^{-i\sigma t}dt \\ -\Big[\frac{1}{2\pi}\int_0^\infty\lambda\frac{\partial}{\partial t}\left\{a^{-1}\Big(1-\frac{M_H^2}{4a^2r^2}\Big)^{-1}\right\}\frac{\partial\delta\omega}{\partial r}\,e^{-i\sigma t}dt+\frac{1}{2\pi}\int_0^\infty\lambda\left\{a^{-1}\Big(1-\frac{M_H^2}{4a^2r^2}\Big)^{-1}\right\}\frac{\partial^2\delta\omega}{\partial r\partial t}\,e^{-i\sigma t}dt \\ -\frac{1}{2\pi}\left\{\int_0^\infty\lambda\frac{\partial\lambda}{\partial t}\frac{\partial\delta q_2}{\partial t}\,e^{-i\sigma t}dt+\int_0^\infty\lambda^2\frac{\partial^2\delta q_2}{\partial t^2}\,e^{-i\sigma t}dt\right\}\Big]\,. \end{aligned} \tag{5.35}$$

Similarly, after converting the partial derivatives of the perturbations w.r.t. $\bar{t}$ into that w.r.t. t, then inserting those in the equation 5.32, and then Fourier-transforming both sides of the equation from time-space to frequency-space, we get :

$$\frac{1}{2\pi}\int_0^\infty \frac{\Delta\lambda}{R^4\,Sin^3\theta}\frac{\partial}{\partial R}(\lambda Q)\;e^{-i\sigma t}dt = -\frac{1}{2\pi}\int_0^\infty \lambda\Delta\,Sin\,\theta\,F\mathcal{R}\,\delta q_3\,e^{-i\sigma t}dt$$
$$+\frac{1}{2\pi}\int_0^\infty\left\{\lambda\frac{\partial^2\delta\omega}{\partial t\partial\theta}+\mathbb{F}\frac{\partial\delta\omega}{\partial\theta}\right\}e^{-i\sigma t}dt+\frac{1}{2\pi}\int_0^\infty\lambda\left\{-\frac{\partial\lambda}{\partial t}\frac{\partial\delta q_3}{\partial t}-\frac{\partial\delta q_3}{\partial t}\mathbb{F}-\lambda\frac{\partial^2\delta q_3}{\partial t^2}\right\}e^{-i\sigma t}dt\,. \tag{5.36}$$

Now, applying the formula regarding Fourier-transformations and results of 'Convolution-theorem' of Fourier-transformation for the product of two functions, which have been shown and derived in the APPENDIX-4 i.e. section 5.9, the equations 5.35 and 5.36 can be written as respectively :

$$\begin{aligned}\left(\frac{F\lambda}{R^4\,Sin^3\theta}\right)^\dagger\frac{\partial Q^\dagger}{\partial\theta}&=\left\{F^2R\,Sin\,\theta\Big(1-\frac{2M_H}{R}\Big)^{\frac12}\Big(\frac{\lambda}{F}\Big)^{\frac12}a^{-1}\Big(1-\frac{M_H^2}{4a^2r^2}\Big)^{-1}\mathcal{R}\right\}^\dagger\delta q_2^\dagger\\
&-\left\{a^{-1}\Big(1-\frac{M_H^2}{4a^2r^2}\Big)^{-1}\mathscr{F}\right\}^\dagger\frac{\partial\delta\omega^\dagger}{\partial r}+(\lambda\mathscr{F})^\dagger\,(i\sigma\delta q_2)^\dagger\\
&-\Big[\left\{\lambda\frac{\partial}{\partial t}\Big(a\Big(1-\frac{M_H^2}{4a^2r^2}\Big)\Big)^{-1}\right\}^\dagger\frac{\partial\delta\omega^\dagger}{\partial r}+\left\{\lambda\Big(a\Big(1-\frac{M_H^2}{4a^2r^2}\Big)\Big)^{-1}\right\}^\dagger i\sigma\frac{\partial\delta\omega^\dagger}{\partial r}\\
&-\left\{\lambda\frac{\partial\lambda}{\partial t}i\sigma+\lambda^2(i\sigma)^2\right\}^\dagger\delta q_2^\dagger\Big]\,,\end{aligned}\tag{5.37}$$

and

$$\begin{aligned}&\left(\frac{\Delta\lambda^2}{R^4\,Sin^3\theta}\right)^\dagger\left(\frac{\partial Q}{\partial R}\right)^\dagger+\left(\frac{\Delta\lambda}{R^4\,Sin^3\theta}\frac{\partial\lambda}{\partial R}\right)^\dagger Q^\dagger\\
&=-\delta q_3^\dagger\,(\lambda\Delta\,Sin\,\theta\,F\mathcal{R})^\dagger+(i\sigma\lambda+\mathbb{F})^\dagger\frac{\partial\delta\omega^\dagger}{\partial\theta}+\left(-i\sigma\lambda\frac{\partial\lambda}{\partial t}-i\sigma\lambda\mathbb{F}+\lambda^2\sigma^2\right)^\dagger\delta q_3^\dagger\,,\end{aligned}\tag{5.38}$$

where the †-superscript over any bracket denotes the Fourier-transform of the quantity within the bracket from time-space to frequency-space.
Now, we have to eliminate $\delta\omega^\dagger$ from these two equations 5.37 and 5.38, so that we may obtain an equation describing the perturbations $\delta q_2^\dagger$ and $\delta q_3^\dagger$ only. For this, we start by partially differentiating these two equations w.r.t. θ and r respectively, so that we may have the quantity $\frac{\partial^2\delta\omega^\dagger}{\partial\theta\partial r}=\frac{\partial^2\delta\omega^\dagger}{\partial r\partial\theta}$ in both the differentiated equations and we can replace that. Hence, partially differentiating both sides of the equation 5.37 w.r.t. θ, we obtain the equation :

$$\begin{aligned}\left(\frac{F\lambda}{R^4}\right)^\dagger\frac{\partial}{\partial\theta}\Big(\frac{1}{Sin^3\theta}\frac{\partial Q^\dagger}{\partial\theta}\Big)&=\left\{F^2R\Big(1-\frac{2M_H}{R}\Big)^{\frac12}\Big(\frac{\lambda}{F}\Big)^{\frac12}\right\}^\dagger\frac{\partial}{\partial\theta}(\delta q_2\mathcal{R}\,Sin\,\theta)^\dagger\\
-\left\{\lambda\frac{\partial}{\partial t}\Big(a^{-1}\Big(1-\frac{M_H^2}{4a^2r^2}\Big)^{-1}\Big)+a^{-1}\Big(1-\frac{M_H^2}{4a^2r^2}\Big)^{-1}\mathscr{F}+i\sigma\lambda a^{-1}\Big(1-\frac{M_H^2}{4a^2r^2}\Big)^{-1}\right\}^\dagger&\frac{\partial^2\delta\omega^\dagger}{\partial\theta\partial r}\\
+\left\{-\lambda^2\sigma^2+i\sigma\left(\lambda\frac{\partial\lambda}{\partial t}+\mathscr{F}\lambda\right)\right\}^\dagger&\frac{\partial\delta q_2^\dagger}{\partial\theta}\,.\end{aligned}\tag{5.39}$$

Next, partially differentiating both sides of the equation 5.38 w.r.t. r, we obtain :

$$\frac{1}{Sin^3\theta}\frac{\partial}{\partial r}\left[\left(\frac{\Delta\lambda}{R^4}\right)^{\dagger}\left\{\frac{\partial}{\partial R}(\lambda Q)\right\}^{\dagger}\right] = -Sin\,\theta\,\frac{\partial}{\partial r}\left\{\delta q_3^{\dagger}(\lambda\Delta F\mathcal{R})^{\dagger}\right\}$$
$$+(i\sigma\lambda+\mathbb{F})^{\dagger}\frac{\partial^2\delta\omega^{\dagger}}{\partial r\partial\theta}+\frac{\partial\delta\omega^{\dagger}}{\partial\theta}\frac{\partial}{\partial r}(i\sigma\lambda+\mathbb{F})^{\dagger}+\left(-i\sigma\lambda\frac{\partial\lambda}{\partial t}-i\sigma\lambda\mathbb{F}+\lambda^2\sigma^2\right)^{\dagger}\frac{\partial\delta q_3^{\dagger}}{\partial r} \tag{5.40}$$
$$+\delta q_3^{\dagger}\frac{\partial}{\partial r}\left\{\lambda\left(-i\sigma\frac{\partial\lambda}{\partial t}-i\sigma\mathbb{F}+\lambda\sigma^2\right)\right\}^{\dagger}.$$

Again, for brevity, we designate the quantity $a\left(1-\frac{M_H^2}{4a^2r^2}\right)\frac{\partial}{\partial t}\left(a\left(1-\frac{M_H^2}{4a^2r^2}\right)\right)^{-1}$ as $\mathcal{F}$. Now we have to substitute the quantity $\frac{\partial^2\delta\omega^{\dagger}}{\partial\theta\partial r}$ from the equation 5.39 in the equation 5.40. But, the equation, which is obtained after substituting the expression of $\frac{\partial^2\delta\omega^{\dagger}}{\partial\theta\partial r}$ from the equation 5.39 in the equation 5.40, would have a term containing $\frac{\partial\delta\omega^{\dagger}}{\partial\theta}$. Therefore, in that equation, we have to again substitute the $\frac{\partial\delta\omega^{\dagger}}{\partial\theta}$ from the equation 5.38, to make in completely in-terms of the perturbations $\delta q_3^{\dagger}$ and $\delta q_2^{\dagger}$. Doing these substitutions, we obtain :

$$\frac{1}{Sin^3\theta}\frac{\partial}{\partial r}\left[\left(\frac{\Delta\lambda}{R^4}\right)^{\dagger}\left\{\frac{\partial}{\partial R}(\lambda Q)\right\}^{\dagger}\right] =$$
$$-\frac{(i\sigma\lambda+\mathbb{F})^{\dagger}}{\left\{(i\sigma\lambda+\mathscr{F}+\mathcal{F})\left(a\left(1-\frac{M_H^2}{4a^2r^2}\right)\right)^{-1}\right\}^{\dagger}}\left[\left\{\left(\frac{F\lambda}{R^4}\right)^{\dagger}\frac{1}{Sin^3\theta}\left(-3Cot\,\theta\frac{\partial Q^{\dagger}}{\partial\theta}+\frac{\partial^2Q^{\dagger}}{\partial\theta^2}\right)\right\}\right.$$
$$\left.-\left\{F^2R\left(1-\frac{2M_H}{R}\right)^{\frac{1}{2}}\left(\frac{\lambda}{F}\right)^{\frac{1}{2}}\right\}^{\dagger}\frac{\partial}{\partial\theta}(\delta q_2\mathcal{R}\,Sin\,\theta)^{\dagger}\right]-Sin\,\theta\frac{\partial}{\partial r}\left\{\delta q_3^{\dagger}(\lambda\Delta F\mathcal{R})^{\dagger}\right\}$$
$$+\left\{\lambda\left(\lambda\sigma^2-i\sigma\frac{\partial\lambda}{\partial t}\right)a\left(1-\frac{M_H^2}{4a^2r^2}\right)\right\}^{\dagger}\left\{\left(\frac{\partial\delta q_3}{\partial R}\right)^{\dagger}-\frac{(i\sigma\lambda+\mathbb{F})^{\dagger}}{(i\sigma\lambda+\mathscr{F}+\mathcal{F})^{\dagger}}\left(\frac{\partial\delta q_2}{\partial\theta}\right)^{\dagger}\right\}$$
$$+(i\sigma\lambda)^{\dagger}\left[\frac{(i\sigma\lambda+\mathbb{F})^{\dagger}}{\left\{(i\sigma\lambda+\mathscr{F}+\mathcal{F})\left(a\left(1-\frac{M_H^2}{4a^2r^2}\right)\right)^{-1}\right\}^{\dagger}}\left(\mathscr{F}\left(\frac{\partial\delta q_2}{\partial\theta}\right)\right)^{\dagger}-\left\{\frac{\mathbb{F}}{\left(a\left(1-\frac{M_H^2}{4a^2r^2}\right)\right)^{-1}}\left(\frac{\partial\delta q_3}{\partial R}\right)\right\}^{\dagger}\right]$$
$$+\frac{\frac{\partial}{\partial r}(i\sigma\lambda+\mathbb{F})^{\dagger}}{(i\sigma\lambda+\mathbb{F})^{\dagger}}\left\{\left(\frac{\Delta\lambda}{R^4\,Sin^3\theta}\right)^{\dagger}\left(\frac{\partial}{\partial R}(\lambda Q)\right)^{\dagger}+\delta q_3^{\dagger}(\lambda\Delta\,Sin\,\theta F\mathcal{R})^{\dagger}\right\}$$
$$-\left\{i\sigma\lambda\left(\frac{\partial\lambda}{\partial t}+\mathbb{F}+i\sigma\lambda\right)\right\}^{\dagger}\left[-\frac{\frac{\partial}{\partial r}\left\{-i\sigma\lambda\left(\frac{\partial\lambda}{\partial t}+\mathbb{F}+i\sigma\lambda\right)\right\}^{\dagger}}{\left\{-i\sigma\lambda\left(\frac{\partial\lambda}{\partial t}+\mathbb{F}+i\sigma\lambda\right)\right\}^{\dagger}}+\frac{\frac{\partial}{\partial r}(i\sigma\lambda+\mathbb{F})^{\dagger}}{(i\sigma\lambda+\mathbb{F})^{\dagger}}\right]\delta q_3^{\dagger}. \tag{5.41}$$

Some simplification is required before further proceeding with the equation 5.41. It is to be noted that some terms on the RHS of the equation 5.41 contain the Ricci scalar $\mathcal{R}$, multiplied with other quantities, while in a Fourier-transformed form and the whole function where it appears is acted by partial derivative w.r.t. r or θ.

We check the value of the Ricci scalar in this regard. Now, we show that at any arbitrary distance from a PBH, described by any arbitrary spherically symmetric and diagonal metric, the Ricci scalar vanishes when the cosmic fluid is radiation i.e. has the equation-of-state parameter 1/3. This can be shown using the Einstein's equation in the following way. The Einstein's equation gives :

$$R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = (8\pi)T_{\mu\nu}\,. \tag{5.42}$$

Contracting both sides of the above equation 5.42 with $g^{\mu\nu}$, we get

$$\mathcal{R} - \frac{4}{2}\mathcal{R} = -\mathcal{R} = (8\pi)T_{\mu\nu}g^{\mu\nu}\,. \tag{5.43}$$

Now, we substitute the stress-energy tensor component for an imperfect fluid :

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} + (\gamma_\mu u_\nu + \gamma_\nu u_\mu) + \Pi_{\mu\nu}\,, \tag{5.44}$$

in the RHS of the above equation 5.43. Here, γ_μ is the heat-flux vector and $\Pi_{\mu\nu}$ is the viscous-shear tensor for the concerned fluid ; while u_μ denotes the four-velocity of the fluid. Then, on the RHS of the equation 5.43, we get, after contracting $T_{\mu\nu}$ with $g^{\mu\nu}$:

$$\begin{aligned} T_{\mu\nu}g^{\mu\nu} &= (\rho+p)u_\mu u_\nu g^{\mu\nu} + pg_{\mu\nu}g^{\mu\nu} + (\gamma_\mu u_\nu + \gamma_\nu u_\mu)g^{\mu\nu} + \Pi_{\mu\nu}g^{\mu\nu} \\ &= -(\rho+p) + 4p + (\gamma_\mu u^\mu + \gamma_\nu u^\nu) + \Pi_{\mu\nu}g^{\mu\nu} \\ &= -\rho + 3p + 2\gamma_\mu u^\mu + \Pi_{\mu\nu}g^{\mu\nu}\,. \end{aligned} \tag{5.45}$$

In our case the concerned cosmic fluid is radiation, then $p = \rho/3$. Again, as the heat-flux vector is transverse to the world-lines, $\gamma_\mu u^\mu = 0$. Using these finally we obtain,

$$T_{\mu\nu}g^{\mu\nu} = \Pi_{\mu\nu}g^{\mu\nu}\,. \tag{5.46}$$

Therefore, if we assume that the radiation is viscosity free or $\Pi_{\mu\nu} = 0$, then $T_{\mu\nu}g^{\mu\nu} = 0$. Then, the equation 5.43 gives $\mathcal{R} = 0$. Now, using the fact that, in the case of our interest the Ricci scalar vanishes everywhere around the black hole, the equation 5.41 simplifies to :

$$\begin{aligned} &\frac{1}{Sin^3\theta}\frac{\partial}{\partial r}\left[\left(\frac{\Delta\lambda}{R^4}\right)^\dagger\left\{\frac{\partial}{\partial R}(\lambda Q)\right\}^\dagger\right] = \\ &-\frac{(i\sigma\lambda+\mathbb{F})^\dagger}{\left\{(i\sigma\lambda+\mathscr{F}+\mathcal{F})\left(a\left(1-\frac{M_H^2}{4a^2r^2}\right)\right)^{-1}\right\}^\dagger}\left[\left\{\left(\frac{F\lambda}{R^4}\right)^\dagger\frac{1}{Sin^3\theta}\left(-3Cot\,\theta\frac{\partial Q^\dagger}{\partial\theta}+\frac{\partial^2 Q^\dagger}{\partial\theta^2}\right)\right\}\right. \\ &+\left\{\lambda\left(\lambda\sigma^2 - i\sigma\frac{\partial\lambda}{\partial t}\right)a\left(1-\frac{M_H^2}{4a^2r^2}\right)\right\}^\dagger\left\{\left(\frac{\partial\delta q_3}{\partial R}\right)^\dagger - \frac{(i\sigma\lambda+\mathbb{F})^\dagger}{(i\sigma\lambda+\mathscr{F}+\mathcal{F})^\dagger}\left(\frac{\partial\delta q_2}{\partial\theta}\right)^\dagger\right\} \\ &+\left\{i\sigma\lambda\mathbb{F}\left(a\left(1-\frac{M_H^2}{4a^2r^2}\right)\right)\right\}^\dagger\left[\frac{(i\sigma\lambda+\mathbb{F})^\dagger}{(i\sigma\lambda+\mathscr{F}+\mathcal{F})^\dagger}\left\{\frac{\mathscr{F}}{\mathbb{F}}\left(\frac{\partial\delta q_2}{\partial\theta}\right)\right\}^\dagger - \left(\frac{\partial\delta q_3}{\partial R}\right)^\dagger\right] \\ &+\frac{\frac{\partial}{\partial r}(i\sigma\lambda+\mathbb{F})^\dagger}{(i\sigma\lambda+\mathbb{F})^\dagger}\left[\left(\frac{\Delta\lambda}{R^4\,Sin^3\theta}\right)^\dagger\left\{\frac{\partial}{\partial R}(\lambda Q)\right\}^\dagger\right] \\ &-\left\{i\sigma\lambda\left(\frac{\partial\lambda}{\partial t}+\mathbb{F}+i\sigma\lambda\right)\right\}^\dagger\left[-\frac{\frac{\partial}{\partial r}\left\{i\sigma\lambda\left(\frac{\partial\lambda}{\partial t}+\mathbb{F}+i\sigma\lambda\right)\right\}^\dagger}{\left\{i\sigma\lambda\left(\frac{\partial\lambda}{\partial t}+\mathbb{F}+i\sigma\lambda\right)\right\}^\dagger}+\frac{\frac{\partial}{\partial r}(i\sigma\lambda+\mathbb{F})^\dagger}{(i\sigma\lambda+\mathbb{F})^\dagger}\right]\delta q_3^\dagger\,. \end{aligned} \tag{5.47}$$

At this stage, we note that the above equation 5.47 can yet not be expressed completely in terms of the perturbation variable $Q^\dagger$. But, we see that after applying certain approximations, the above equation 5.47 can be expressed w.r.t. $Q^\dagger$ only and subsequently the separation of variables technique can be applied to it. We describe this in the next section.

5.4 The simplified equation after applying the approximations and separation of variables

5.4.1 Separation of Radial and Angular Parts of the equation :

After applying the approximations described in the APPENDIX-5 i.e. section 5.10, including the approximation $\frac{(i\sigma\lambda+\mathscr{F})^\dagger}{(i\sigma\lambda+\mathscr{F}+\mathcal{F})^\dagger} \approx 1$, and expressing the quantities $Q^\dagger_{23}$ in terms of $Q^\dagger$, the equation 5.47 can be written as :

$$\left(\frac{\partial}{\partial R}\right)^\dagger \left[\left(\frac{\Delta\lambda}{R^4}\right)^\dagger \left\{\frac{\partial}{\partial R}(\lambda Q)\right\}^\dagger\right] = -\left(\frac{F\lambda}{R^4}\right)^\dagger \Big(-3Cot\,\theta \frac{\partial Q^\dagger}{\partial\theta} + \frac{\partial^2 Q^\dagger}{\partial\theta^2}\Big)$$
$$+\left\{\frac{\lambda}{\Delta}\left(\lambda\sigma^2 - i\sigma\frac{\partial\lambda}{\partial t}\right)\right\}^\dagger (-Q^\dagger) + \left(i\sigma\frac{\lambda\mathscr{F}}{\Delta}\right)^\dagger Q^\dagger + \frac{\left(\frac{\partial}{\partial R}\right)^\dagger (i\sigma\lambda+\mathbb{F})^\dagger}{(i\sigma\lambda+\mathbb{F})^\dagger}\left(\frac{\Delta\lambda}{R^4}\right)^\dagger \left\{\frac{\partial}{\partial R}(\lambda Q)\right\}^\dagger . \tag{5.48}$$

Thereafter we express the quantity $Q^\dagger$ as the multiplication of two parts $Q^\dagger_R(r,\sigma)$ and $Q^\dagger_\theta(\theta)$ as $Q^\dagger = Q^\dagger_R Q^\dagger_\theta$, where the part $Q^\dagger_R$, called the radial part, is a function of r and σ ; while the part $Q^\dagger_\theta$, called the angular part, is a function of θ. [3] We now substitute $Q^\dagger = Q^\dagger_R Q^\dagger_\theta$ in the equation 5.48, to implement the procedure known as separation of variables technique and thus we obtain :

$$\frac{1}{Q^\dagger_R}\left(\frac{R^4}{F\lambda}\frac{\partial}{\partial R}\right)^\dagger \left[\left(\frac{\Delta\lambda}{R^4}\right)^\dagger \left\{\frac{\partial}{\partial R}(\lambda Q_R)\right\}^\dagger\right] + \left(\frac{R^4}{F\Delta}\right)^\dagger \left\{\left(\lambda\sigma^2 - i\sigma\frac{\partial\lambda}{\partial t}\right) - i\sigma\mathscr{F}\right\}^\dagger$$
$$-\frac{1}{Q^\dagger_R}\frac{\left(\frac{\partial}{\partial R}\right)^\dagger (i\sigma\lambda+\mathbb{F})^\dagger}{(i\sigma\lambda+\mathbb{F})^\dagger}\left(\frac{\Delta}{F}\right)^\dagger \left\{\frac{\partial}{\partial R}(\lambda Q_R)\right\}^\dagger = -\frac{1}{Q^\dagger_\theta}\Big(-3Cot\,\theta\frac{\partial Q^\dagger_\theta}{\partial\theta} + \frac{\partial^2 Q^\dagger_\theta}{\partial\theta^2}\Big). \tag{5.49}$$

Now, in the above equation 5.49, the LHS is a function of r and t, while the RHS is a function of θ. Therefore, we can say that in the above equation 5.49, the LHS = RHS = constant, which is independent of r, t and θ. Let this constant be $\mathbb{K}$. Then, we can write two equations resulting from the equation 5.49 as :

$$-\frac{1}{Q^\dagger_\theta}\Big(-3Cot\theta\frac{\partial Q^\dagger_\theta}{\partial\theta} + \frac{\partial^2 Q^\dagger_\theta}{\partial\theta^2}\Big) = \mathbb{K} \tag{5.50}$$

and

[3] It is to be noted that, as we had inserted the fourier-transformed axial-perturbations into the equations in Section 5.3, from then their derivative w.r.t. isotropic time t is carried by multiplication with a factor of $i\sigma$ in fourier-space. The fourier-transformed axial-perturbations are independent on the isotropic time t, and is depedent on isotropic radial coordinate r, angular coordinate θ and frequency σ.

$$\frac{1}{Q_R^\dagger}\left(\frac{R^4}{F\lambda}\frac{\partial}{\partial R}\right)^\dagger\left[\left(\frac{\Delta\lambda}{R^4}\right)^\dagger\left\{\frac{\partial}{\partial R}(\lambda Q_R)\right\}^\dagger\right]+\left(\frac{R^4}{F\Delta}\right)^\dagger\left\{\left(\lambda\sigma^2-i\sigma\frac{\partial\lambda}{\partial t}\right)-i\sigma\mathscr{F}\right\}^\dagger$$
$$-\frac{1}{Q_R^\dagger}\frac{\left(\frac{\partial}{\partial R}\right)^\dagger(i\sigma\lambda+\mathbb{F})^\dagger}{(i\sigma\lambda+\mathbb{F})^\dagger}\left(\frac{\Delta}{F}\right)^\dagger\left\{\frac{\partial}{\partial R}(\lambda Q_R)\right\}^\dagger=\mathbb{K}\,. \tag{5.51}$$

So, we see that the equation satisfied by the angular part $Q_\theta^\dagger$ i.e. the equation 5.50, is same as that is obtained in case of the Schwarzschild metric, as expected. Hence, the angular part in this case too, like the Schwarzschild metric, can be taken as : $Q_\theta^\dagger \propto \mathcal{C}_{l+2}^{-3/2}$ [191]; where $\mathcal{C}_n^\nu$ is the 'Gegenbauer function'. This function $\mathcal{C}_n^\nu$ satisfies the equation :

$$\left[\frac{d}{d\theta}Sin^{2\nu}\theta\frac{d}{d\theta}+n(n+2\nu)Sin^{2\nu}\theta\right]\mathcal{C}_n^\nu=0\,. \tag{5.52}$$

It may be noted that the 'Gegenbauer function' $\mathcal{C}_{l+2}^{-3/2}$, associated with this case is related to the 'Legendre Polynomial function' $P_l(\theta)$ by the formulae :

$$\mathcal{C}_{l+2}^{-3/2}(\theta)=Sin^3\theta\frac{d}{d\theta}\Big(\frac{1}{Sin\theta}\frac{d}{d\theta}P_l(\theta)\Big)\,. \tag{5.53}$$

The angular part $Q_\theta^\dagger$ being proportional to the $\mathcal{C}_{l+2}^{-3/2}$, sets the constant $\mathbb{K}$ and hence, the equation satisfied by the radial part $Q_R^\dagger$ becomes :

$$\left(\frac{R^4}{F\lambda}\frac{\partial}{\partial R}\right)^\dagger\left[\left(\frac{\Delta\lambda}{R^4}\right)^\dagger\left\{\frac{\partial}{\partial R}(\lambda Q_R)\right\}^\dagger\right]+\left(\frac{R^4}{F\Delta}\right)^\dagger\left\{\left(\lambda\sigma^2-i\sigma\frac{\partial\lambda}{\partial t}\right)-i\sigma\mathscr{F}\right\}^\dagger Q_R^\dagger$$
$$-\frac{\left(\frac{\partial}{\partial R}\right)^\dagger(i\sigma\lambda+\mathbb{F})^\dagger}{(i\sigma\lambda+\mathbb{F})^\dagger}\left(\frac{\Delta}{F}\right)^\dagger\left\{\frac{\partial}{\partial R}(\lambda Q_R)\right\}^\dagger=\mathscr{M}^2Q_R^\dagger\,, \tag{5.54}$$

where the quantity $\mathscr{M}=(l+2)(l-1)$, where l is a positive integer $\geqslant 2$, describes the angular dependence.

Let us see to which form the above equation 5.54 reduces if we go from generalized McVittie metric to Schwarzschild metric of a non-rotating uncharged black hole of constant mass. We denote the radial coordinate and the constant mass of the Schwarzschild black hole as r and m respectively, while the perturbation-variable can be represented as simply Q_R, because its fourier-transformation is not required in the case of time-independent Schwarzschild metric. In case of Schwarzschild black hole of constant mass, the quantity λ becomes 1 (C becomes 0 and F becomes 1). Also, the quantities $\mathbb{F}$ and $\mathscr{F}$ become zero, as these involve derivatives w.r.t. time. Hence, the equation 5.54 reduces to :

$$\Delta\frac{d}{dr}\left(\frac{\Delta}{r^4}\frac{dQ_R}{dr}\right)+\sigma^2Q_R-\mathscr{M}^2\frac{\Delta}{r^4}Q_R=0\,, \tag{5.55}$$

where the partial derivatives have been replaced by total derivatives w.r.t. r, as for Schwarzschild metric of constant mass the concerned quantities are time-independent.

The equation 5.55 is just another-form of the 'Regge-Wheeler equation'. Therefore, we get the Regge-Wheeler equation from the equation 5.54, when generalized McVittie metric reduces to time-independent Schwarzschild metric. This in fact proves one facet of the correctness of the equation 5.54.

5.4.2 The horizons in the generalized McVittie metric :

At this stage it is necessary to introduce the horizons of the black hole. The horizons of the generalized McVittie metric are given by the condition :

$$\left(1 - \frac{C^2}{A^2}\right) = 0 \Rightarrow A^2 = C^2 \,. \tag{5.56}$$

The expression of these horizons have been derived in the reference [192] and these are given by :

$$R_{\pm} = \frac{1}{2H}\left[1 - M_H\left(1 + \frac{m}{2r}\right)\frac{\dot{m}}{m} \pm \sqrt{\left\{1 - M_H\left(1 + \frac{m}{2r}\right)\frac{\dot{m}}{m}\right\}^2 - 8m\dot{a}}\,\right] \,, \tag{5.57}$$

where it is to be noted that as R depends on r, the above expression of $R_{\pm}$ is implicit. Here, when the quantity inside the square-root i.e. $\left\{1 - M_H\left(1 + \frac{m}{2r}\right)\frac{\dot{m}}{m}\right\}^2 - 8m\dot{a}$ is positive, the horizons are physical. Then, the larger R_+ can be called the 'Cosmic apparent horizon' and the smaller R_- can be called the 'Black hole apparent horizon'. It is quite clear that when the quantity inside the square-root on the RHS of equation 5.57 vanishes then these two horizons coincide and when this quantity is negative, there is no physical horizon, instead 'naked singularity' comes there. The later two cases of coinciding horizons and 'naked singularity' are not of interest in our case.
Therefore in this scenario, whenever, we have to deal with a description of the perturbations or any function of perturbations in this space-time, we have to confine that from the black hole apparent horizon to the cosmic apparent horizon.

5.4.3 Change of variables and the boundary conditions :

Now, we shall transform the radial coordinate from R to $R_\star$, where the new coordinate $R_\star$ is defined as [4] :

$$\frac{\partial}{\partial R_\star} \equiv \left(\frac{\Delta}{R^2}\frac{\partial}{\partial R}\right)^{\dagger} \,. \tag{5.58}$$

Again, for convenience, we define a new variable $\tilde{Q}_R = (\lambda Q_R)^{\dagger}$. The interesting issue about this variable is that as $\left(1 - \frac{C^2}{A^2}\right) = 0$ at the horizons $R_{\pm}$, this variable vanishes at both the horizons, except for zero frequency modes. Also, it should be mentioned that although the boundary conditions for $Q_R^{\dagger}$ has to be purely ingoing at the black hole apparent horizon R_- and purely outgoing at the cosmic apparent horizon R_+, that does not affect the vanishing of the new variable $\tilde{Q}_R$ at both the horizons.
Using the new radial coordinate $R_\star$ and the new variable $\tilde{Q}_R$, putting $\mathscr{F}$ in place of $\mathbb{F}$ (as $\mathbb{F} \approx \mathscr{F}$,

[4] It is to be noted that $R_\star$ is not a 'tortoise-coordinate' in this case.

the approximation which we have already applied), the above equation 5.54 can be written as :

$$\frac{\partial^2 \tilde{Q}_R}{\partial R_\star^2} + \left[\frac{\frac{\partial}{\partial R_\star}\left(\frac{\lambda}{R^2}\right)^\dagger}{\left(\frac{\lambda}{R^2}\right)^\dagger} - \frac{\frac{\partial}{\partial R_\star}(i\sigma\lambda + \mathscr{F})^\dagger}{(i\sigma\lambda + \mathscr{F})^\dagger} \right] \frac{\partial \tilde{Q}_R}{\partial R_\star} + \left\{ \sigma^2 - i\sigma \left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial t} + \frac{\mathscr{F}}{\lambda} \right)^\dagger \right\} \tilde{Q}_R = \mathscr{M}^2 \left\{ \frac{\Delta}{R^4}\left(\frac{\lambda}{F}\right)^{-1} \right\}^\dagger \tilde{Q}_R^\dagger . \tag{5.59}$$

Sometimes it may be necessary to make the equation 5.59 free of the single-derivative term. So, we briefly describe the transformation of the equation 5.59 into the form containing only second-order derivative term. The process of elimination of first-order derivative term, from a general second-order differential-equation, is quite well-known. Let us first write the equation 5.59 with the following notations for brevity :

$$\frac{\partial^2 \tilde{Q}_R}{\partial R_\star^2} + \xi(r,\sigma)\frac{\partial \tilde{Q}_R}{\partial R_\star} + \zeta(r,\sigma)\tilde{Q}_R = 0 \, , \tag{5.60}$$

where the quantities $\xi(r,\sigma)$ and $\zeta(r,\sigma)$ stand for respectively :

$$\xi(r,\sigma) = \left[\frac{\frac{\partial}{\partial R_\star}\left(\frac{\lambda}{R^2}\right)^\dagger}{\left(\frac{\lambda}{R^2}\right)^\dagger} - \frac{\frac{\partial}{\partial R_\star}(i\sigma\lambda + \mathscr{F})^\dagger}{(i\sigma\lambda + \mathscr{F})^\dagger} \right] , \tag{5.61}$$

and

$$\zeta(r,\sigma) = \left\{ \sigma^2 - i\sigma \left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial t} + \frac{\mathscr{F}}{\lambda} \right)^\dagger \right\} - \mathscr{M}^2 \left\{ \frac{\Delta}{R^4}\left(\frac{\lambda}{F}\right)^{-1} \right\}^\dagger . \tag{5.62}$$

Now, we use the substitution :

$$\Psi_R = exp. \left\{ \int^{R_\star} \xi \, dR' \right\} \tilde{Q}_R(R_\star) \tag{5.63}$$

in the equation 5.60. With this substitution, after some calculations the equation 5.60 is transformed into the form :

$$\frac{\partial^2 \Psi_R}{\partial R_\star^2} + \left\{ \zeta - \frac{1}{2}\Big(\frac{\partial \xi}{\partial R_\star}\Big) - \frac{1}{4}\xi^2 \right\} \Psi_R = 0 \, . \tag{5.64}$$

The above equation 5.64 may be called the *equivalent-counterpart of the 'Regge-Wheeler equation' for generalized McVittie metric* : the equation governing the 'Axial perturbations' in the space-time of a Schwarzschild black hole embedded in FLRW-Universe, described by the generalized McVittie metric. The equation 5.59 has now been transformed into the equation 5.64, which is a Schrödinger-like form and we rewrite this in the following style :

$$\frac{\partial^2 \Psi_R}{\partial R_\star^2} + \sigma^2 \Psi_R = \left[i\sigma \left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial t} + \frac{\mathscr{F}}{\lambda} \right)^\dagger + \mathscr{M}^2 \left\{ \frac{\Delta}{R^4}\left(\frac{\lambda}{F}\right)^{-1} \right\}^\dagger + \left\{ \frac{1}{2}\Big(\frac{\partial \xi}{\partial R_\star}\Big) + \frac{1}{4}\xi^2 \right\} \right] \Psi_R . \tag{5.65}$$

5.4.4 Some Physical interpretations from the Potential :

From the equation 5.65 in Schrödinger-like form, we can say that the effective potential for Ψ_R is given by :

$$V = \left[i\sigma \left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial t} + \frac{\mathscr{F}}{\lambda} \right)^{\dagger} + \mathscr{M}^2 \left\{ \frac{\Delta}{R^4} \left(\frac{\lambda}{F} \right)^{-1} \right\}^{\dagger} + \left\{ \frac{1}{2} \left(\frac{\partial\xi}{\partial R_\star} \right) + \frac{1}{4}\xi^2 \right\} \right], \tag{5.66}$$

which is a function of both the radial coordinate and frequency.
Some important characteristics of this potential can be noted. Before going into details, we need to describe certain properties of the quantities R, λ, Δ, F etc., in context of their transformation due to sign-reversal of time coordinate. It is to be noted that some of these quantities and their certain functions are even w.r.t. time coordinates t and $\bar{t}$, which can be examined by investigating their transformation under a sign-reversal of these time coordinates (i.e. $t \to -t$, $\bar{t} \to -\bar{t}$).
First of all, we note the property of the quantity $C = HR + \dot{m}a\sqrt{\tilde{r}/r}$. It is evident that R, m, and a must be even w.r.t. t. This is because radial distances R, $\tilde{r}$ or r and mass m can not be negative. A negative scale factor a is also unphysical according to the FLRW-metric. So, the only quantities that change sign under the transformation $t \to -t$ are H and $\dot{m}$, as these contain the derivative w.r.t. t. So, under the transformation $t \to -t$, C changes sign viz. $C \to -C$, or, C is an odd-function of t. Again, the quantity $A = 1 - 2M_H/R$ clearly does not change sign under the transformation $t \to -t$, as it is made up of the parameters a, m, and r. Similarly, the quantity $\Delta = R^2 - 2M_H R$ is also even w.r.t, t.
So, the quantity $(1 - C^2/A^2)$ is even under sign-reversal of t (due to the squared appearance of the quantity C).
We note the equation 5.82 governing the integrating factor F, which is given in Appendix-3 i.e. section 5.8. The quantity β on the RHS of the equation 5.82, is clearly odd under the sign-reversal of t, as its denominator $A^2 - C^2$ is even and numerator C is odd under the transformation $t \to -t$. Now, if we denote the solution of this equation as F' for the sign-reversed case of $t \to -t$, then it is simple to check that the equation for F' and F will be same. Hence, F should be unchanged under the transformation $t \to -t$ or, F is an even function under the sign-reversal of t. As F and $(1 - C^2/A^2)$ both are even under the transformation $t \to -t$, the quantity $\lambda = F\,(1 - C^2/A^2)$ is also even under sign-reversal of t.
Another fact is pertinent to mention here, although it is quite clear from the above discussion, for the line-element in 'Nolan-gauge' (Schwarzschild-like coordinates) in 5.9, from the definition of $\bar{t}$, it follows that for a sign-reversal in t implies a similar transformation in $\bar{t}$ i.e. $t \to -t \Rightarrow \bar{t} \to -\bar{t}$.
Again, the quantity $\mathscr{F} = \frac{F}{R^4}\frac{\partial}{\partial \bar{t}}\frac{R^4}{F}$ is an odd function under sign-reversal of time t or $\bar{t}$.
Therefore from the above analysis, we see that in the potential V, the quantity $\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial t} + \frac{\mathscr{F}}{\lambda} \right)$ is an odd function of time t and the quantity $\left\{ \frac{\Delta}{R^4} \left(\frac{\lambda}{F} \right)^{-1} \right\}$ is an even function of time t. So, the fourier-transforms of the quantities $\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial t} + \frac{\mathscr{F}}{\lambda} \right)$ and $\left\{ \frac{\Delta}{R^4} \left(\frac{\lambda}{F} \right)^{-1} \right\}$ are completely an imaginary quantity and completely a real quantity respectively. In the quantity ξ, λ/R^2 is an even function of time t and hence the first part $\left(\frac{\lambda}{R^2}\right)^{\dagger^{-1}} \frac{\partial}{\partial R_\star} \left(\frac{\lambda}{R^2}\right)^{\dagger}$ within ξ is completely a real quantity. In the second part of ξ, $\mathscr{F}$ is an odd function and λ is an even function of time t.
Hence, it is quite clear that if the frequency σ is a real quantity, then the the overall potential V

given in equation 5.66 is a completely real quantity. [5] This indicates the fact that the potential would be complex, only if the frequency σ is complex quantity. Or, in other way it can be said that the imaginary part of the frequency σ is responsible for the imaginary part of the potential. As a consequence, the stability of the system i.e. whether certain modes of axial perturbations are unstable i.e. if $\sigma_I < 0$ (viz. when the imaginary part of the frequency σ_I is negative) [6] can be examined by styding the imaginary part of the potential.
Also, the contributions of the changing mass of the PBH, surrounding high-density radiation around the PBH and the background expansion of the FLRW-Universe in this potential can be obtained to analyze their relative significance.

5.5 Conclusion and Discussion

In this work, we have derived the equation governing the axial perturbations in the generalized McVittie metric, which can well describe the space-time around a non-rotating uncharged PBH, created in the early radiation-dominated Universe, where the effect of expansion of the Universe on the local space-time of the PBH was significant due to very high value of Hubble-parameter at that time and the PBHs were continuously changing masses due to spherical accretion of the surrounding high-density radiation.
In the process of deriving the equation in desired form, we have done Fourier-transformation of the overall perturbation equations from time-space to frequency-space and then applied several outcomes of the 'Convolution-theorem' for Fourier transform of product of two functions. This procedure of Fourier-transforming the overall perturbation equations, whence not only the perturbation variables but also the unperturbed parameters are time-varying, and then application of the 'Convolution-theorem', to get the equation in a suitable form, should be applicable for deriving equations governing perturbations in case of any time-dependent metric. It may be noted that generally in case of perturbations in any time-dependent black hole metric, either the temporal-part is separated from the spatial part or suitably chosen ansatz is applied for getting the equation in preferable format (for example see the reference [208].).
From the equation governing the axial perturbations in Schrödinger-like form, which is free of first-order derivative term, we see that the potential is a complex-quantity and we have also explained that its imaginary part originates due to the imaginary part of the frequency. So, by examining the imaginary part of the potential we can have important information about the imaginary part of the frequency for any certain mode.
We know that for some of the usual black hole metrics e.g. Schwarzschild, Schwarzschild-Anti-De Sitter etc., the unstable modes do not exist i.e. perturbations, exponentially growing with time, are not practically possible. As the generalized McVittie metric is physically the space-time around a Schwarzschild black hole of varying mass, embedded in an expanding FLRW-Universe, if unstable modes are found to exist, then it can be interpreted that the effects of expansion of the background FLRW-Universe or the time-variation of mass of the black hole are making the existence of the unstable modes possible.
For any unstable mode, as the corresponding axial-perturbation grows exponentially with time, at a certain stage the linear perturbation theory breaks down if the growing perturbation becomes

[5] If σ is real, then $i\sigma\lambda^\dagger$ and $\mathscr{F}^\dagger$ are both completely imaginary quantities and as a result $\frac{\frac{\partial}{\partial R_\star}(i\sigma\lambda+\mathscr{F})^\dagger}{(i\sigma\lambda+\mathscr{F})^\dagger}$ is a completely a real quantity.

[6] Substituting $\sigma = \sigma_R + i\sigma_I$ in the $e^{i\sigma t}$, one gets $e^{i\sigma t} = e^{i(\sigma_R + i\sigma_I)t} = e^{i\sigma_R t}e^{-\sigma_I t}$.

sufficiently large so that it can not be treated by linear perturbation theory. In other way to say, these unstable modes indicate the possibility of non-linear effects in these perturbations. In some earlier works, the possibility of non-linear instabilities have been proposed in fast spinning black holes, with a similarity of turbulence in hydrodynamics [209]. But till date, there is hardly any highly-energetic astrophysical or cosmological phenomena studied in numerical general relativity (e.g. the merging of two black holes in a binary), which can generate observationally important non-linear effects. So in case of the generalized McVittie metric, it will be interesting to investigate if the unstable modes exist, then whether those can give rise to significant non-linear effects. If there exists significant non-linear instability, then that might even leave imprint on the stochastic gravitational wave background produced due to the vibration of perturbed PBHs in the early Universe.

In some cases, non-zero stress-energy surrounding any black hole, which is then called 'dirty black hole', can affect the linear stability of that black hole [210]. This stress-energy can be even due a shell of matter or a planet. So, this indicates a concordance with the case of the generalized McVittie metric, where the mass of the black hole is time-varying due to spherical accretion of the surrounding radiation in the early radiation-dominated era. Here, the surrounding radiation should provide the non-zero stress-energy, which can affect the black hole's linear stability. Although in practical case, there are thought to be many ways to perturb those PBHs, as we have argued in the introduction (section 5.1).

The equation derived by us, is the preliminary step for investigating the conditions of stability or instability of non-rotating uncharged PBHs of changing masses, described by the generalized McVittie metric, in the early radiation-dominated Universe. Though the similar counterpart for polar-perturbations is required too. In future we shall try to investigate the existence of the instability in this case, in terms of various parameters, more specifically.

5.6 APPENDIX 1 : Clarification about some dimensional issues

We have to be clear about the presence of the two fundamental constants G (Universal Gravitational constant) and c (Speed of light in vacuum) in all the expressions, which are generally omitted according to the natural units' convention of taking G,c as 1 (unity). The convention of natural units is okay for purely analytical i.e. non-numerical calculations, but this is not suitable as we need to get exact numerical orders of several quantities.

For the second part of the quantity C i.e. $\dot{m}a\sqrt{\frac{\tilde{r}}{r}}$, we first evaluate $\dot{m}$ in terms of the Hawking-Hayward Quasilocal mass M_H, which is related with the former as $M_H(t) = m(t)a(t)$:

$$\begin{aligned} \dot{m} &= \frac{\dot{M}_H}{a} - \left(\frac{\dot{a}}{a^2}\right)M_H \,, \\ \Rightarrow \dot{m}a &= \dot{M}_H - HM_H \,. \end{aligned} \tag{5.67}$$

To estimate an approximate order of the first term on the RHS in the equation 5.67 i.e. $\dot{M}_H$, we use its expression derived in the reference [190]. This gives the time-rate of change of the Hawking-Hayward Quasi-local mass in terms of cosmic-fluid density at a finite radial distance from the black hole (in isotropic coordinates) :

$$\dot{M}_H = -\frac{G}{2}a\mathcal{B}^2\sqrt{1 + a^2\mathcal{A}^4 u^{r2}}(P(r) + \rho(r))\mathscr{A}u^r \,, \tag{5.68}$$

where the concerned quantities $\mathcal{A}$ and $\mathcal{B}$ have already been defined earlier. u^r is the contra-variant radial component of the four-velocity of cosmic fluid getting spherically accreted by the black hole, $\mathscr{A} = 4\pi\mathcal{A}^4a^2r^2$ is the area of the spherical surface of isotropic radius r, $\rho(r)$ and $P(r)$ are respectively the density and pressure of the cosmic fluid at that isotropic radial distance r.
But, it is easy to verify that this expression of $\dot{M}_H$, when converted to the Schwarzschild-like coordinate in Nolan gauge, it gives :

$$\dot{M}_H = 4\pi c R_-^2(1+w)\rho\,, \tag{5.69}$$

which is simply the time rate of accretion of cosmic-fluid into the black hole, through the apparent black hole horizon R_-. Here, w is the equation-of-state parameter of the cosmic-fluid and as it is radiation in our case, $w = 1/3$.
From the metric given in equation 5.9, it is clear that the quantity $C = HR + \dot{m}a\sqrt{\frac{\tilde{r}}{r}}$, written in the convention of natural units, must be dimensionless. In this context it is also to be noted that the corresponding ratio M_H/R in the metric 5.9, is actually GM_H/c^2R i.e. dimensionless, as the quantity G/c^2 remain omitted when we use the convention of natural units, as has been stated already. Hence, both the quantities HR and $\dot{m}a\sqrt{\frac{\tilde{r}}{r}}$ must be actually dimensionless. We here use the notations viz. [L],[M] and [T] for the dimensions of length, mass and time respectively.
In the second term of C, in $\dot{m}$, including the G/c^2 that occurs with the mass m to make it a length-scale, we get $G\dot{m}/c^2$, which has the dimension of length/time : $[LT^{-1}]$ (Dimension of G : $[G] \equiv [L^3T^{-2}M^{-1}]$). So, although it seems that $[C] \equiv [HR] \equiv [\dot{m}a\sqrt{\frac{\tilde{r}}{r}}] \equiv [LT^{-1}]$, it is not the actual dimension, as C must be dimensionless. So, for being dimensionless there must be a c^{-1} multiplied with these and which is evident as we shall show later that $C = \frac{\partial R}{\partial t} \equiv \frac{\partial R}{c\partial t}$. Therefore, without erasing G and c the actual expression of C is :

$$C(t,R) = \frac{HR}{c} + \frac{G\dot{m}}{c^3}a\sqrt{\frac{\tilde{r}}{r}} \tag{5.70}$$

It can be easily verified that C is dimensionless by substituting the dimensions of corresponding quantities.
The confusion for the expression of $\dot{M}_H$ given in equation 5.68 is deeper if we do not write the omitted G, c in proper places because the author in reference [190] has kept the 'G' in Einstein's equations there, while has omitted the 'c's there and also omitted those 'G','c's present with the masses in the ratios $M_H/r \equiv GM_H/c^2r$ in metric coefficients. It can be easily checked that for $\dot{M}_H$ to be dimensionless there must be Gc^{-3} with it. This can be checked by substituting the dimensions of the corresponding quantities. Thus, keeping the 'G' and 'c's in proper places the actual expressions are :

$$\frac{G\dot{M}_H}{c^3} = -\frac{G}{2c^3}aB^2\sqrt{1 + a^2A^4\left(\frac{u^r}{c}\right)^2}(P(r)+\rho(r))\mathscr{A}u^r\,, \tag{5.71}$$

Or, canceling Gc^{-3} from both sides,

$$\dot{M}_H = -\frac{1}{2}aB^2\sqrt{1 + a^2A^4\left(\frac{u^r}{c}\right)^2}(P(r)+\rho(r))\mathscr{A}u^r\,. \tag{5.72}$$

5.7 APPENDIX 2 : Basic equations describing the Axial perturbations

Here, the origin of the equations governing the perturbations on a metric given by equation 5.10, has been described. In this case, the corresponding unperturbed components of the Ricci tensors are :

$$R_{12} = \frac{1}{2} e^{-2\psi} (e^{-2\nu} e^{-2\mu_3})^{1/2} [((e^{3\psi+\nu-(\mu_2+\mu_3)}) \mathcal{Q}_{32})_{,3} - ((e^{3\psi-\nu-\mu_2+\mu_3}) \mathcal{Q}_{02})_{,0}] \,, \tag{5.73}$$

and

$$R_{13} = \frac{1}{2} e^{-2\psi} (e^{-2\nu} e^{-2\mu_3})^{1/2} [((e^{3\psi+\nu-(\mu_2+\mu_3)}) \mathcal{Q}_{23})_{,2} - ((e^{3\psi-\nu+\mu_2-\mu_3)}) \mathcal{Q}_{03})_{,0}] \,, \tag{5.74}$$

where $\mathcal{Q}_{AB} = q_{A,B} - q_{B,A}$ and $\mathcal{Q}_{A0} = q_{A,0} - \omega_{,A}$ are defined similarly as of Q_{AB} and Q_{A0}, but they contain the unperturbed values of the quantities q_2, q_3 and ω instead of their linear perturbations. As for the metric given in equation 5.10, in our case, $q_2 = q_3 = \omega = 0$, therefore $\mathcal{Q}_{AB} = \mathcal{Q}_{A0} = 0$. Hence, the unperturbed Ricci tensor components $R_{12} = R_{\phi R}$ and $R_{13} = R_{\phi\theta}$ are zero (0).
The origin of the equations governing the perturbations is from Einstein's equations for those components. The Einstein's equation for these components, taken to first order axial perturbations, is given by :

$$R_{ij} + \delta_{\omega,q_2,q_3} R_{ij} - \frac{1}{2}(g_{ij} + \mathcal{R}\delta_{\omega,q_2,q_3} g_{ij} + g_{ij}\delta_{\omega,q_2,q_3}\mathcal{R}) = (8\pi)(T_{ij} + \delta_{\omega,q_2,q_3} T_{ij}) \,, \tag{5.75}$$

where i, j are indices denoting spatial-coordinates. (For avoiding confusion with the radial coordinate R, we denote the Ricci scalar by $\mathcal{R}$.)
Subtracting the unperturbed Einstein's equation from the above equation 5.75 with the linear perturbation, we obtain :

$$\delta_{\omega,q_2,q_3} R_{ij} - \frac{1}{2}(g_{ij}\delta_{\omega,q_2,q_3}\mathcal{R} + \delta_{\omega,q_2,q_3} g_{ij}\mathcal{R}) = (8\pi)\delta_{\omega,q_2,q_3} T_{ij} \,. \tag{5.76}$$

As there is no cross components in the metric, hence for this case i.e. i=1 and j=2,3 ; $g_{12} = g_{13} = 0$, which reduces the above equation to :

$$\delta_{\omega,q_2,q_3} R_{ij} - \frac{1}{2}(\delta_{\omega,q_2,q_3} g_{ij})\mathcal{R} = (8\pi)\delta_{\omega,q_2,q_3} T_{ij} \,. \tag{5.77}$$

We have already shown in the section 5.3 that if the cosmic-fluid is radiation having equation-of-state parameter $w = 1/3$, then the associated Ricci-scalar vanishes. Hence, the equation 5.77 further reduces to :

$$\delta_{\omega,q_2,q_3} R_{ij} = (8\pi)\delta_{\omega,q_2,q_3} T_{ij} \,. \tag{5.78}$$

While it can be shown that in a flat FLRW-Universe, for the part of the energy-momentum tensor belonging to a perfect fluid, the concerned components of the linear perturbations of energy-momentum tensor of the cosmic-fluid are zero : $\delta T_{ij\,p} = 0$, where $i \neq j$ and the suffix 'p' in the energy-momentum tensor component represents it is due to the 'perfect' part of the fluid. But, in this case of generalized McVittie metric, a perfect fluid can not describe the surrounding cosmic-fluid. As it has been already shown and explained in the reference [190] that a single perfect cosmic-fluid can not describe a physical solution of a spherically symmetric black hole embedded in an expanding FLRW-Universe. We need at least one imperfectness parameter in it to describe the solution physically. As was chosen by the authors in reference [190], we also choose this imperfectness parameter to be the heat-flux vector γ_μ. Only one component of the heat-flux vector suffices in this case and we can take it to be the radial component, in accordance with the radial mass-flow into the accreting black hole. So, due to the heat-flux vector there would be an additional imperfect part in the energy-momentum tensor of the cosmic-fluid i.e. radiation in this case, which is $T_{ij\,Ip} = \gamma_i u_j + \gamma_j u_i$ (where 'Ip' represents 'imperfect'). So, the perturbation to this is given by (we write δ in place of δ_{ω,q_2,q_3} for brevity):

$$\delta T_{ij\,Ip} = u_j \delta\gamma_i + \gamma_i \delta u_j + u_i \delta\gamma_j + \gamma_j \delta u_i \,. \tag{5.79}$$

The components, with which we have to deal with are : $\delta T_{\phi R\,Ip}$ and $\delta T_{\phi\theta\,Ip}$. For the component $\delta T_{\phi\theta\,Ip}$, the associated components of the four-velocity $u_\phi\,, u_\theta$ are zero ; and also the associated components of the heat-flux vectors $\gamma_\theta\,, \gamma_\phi$ are zero. Hence, the $\delta T_{\phi\theta\,Ip}$ vanishes. On the other hand, for the component $\delta T_{\phi R\,Ip}$, two among the four terms, which contains u_ϕ and γ_ϕ, vanishes. Hence,

$$\delta T_{\phi R\,Ip} = \gamma_R \delta u_\phi + u_R \delta\gamma_\phi \,. \tag{5.80}$$

As a non-zero velocity perturbation in the ϕ-direction would imply presence of angular momentum in that direction, in the cosmic-fluid being accreted by the black hole, then the accretion would no longer remain spherical and that would result in the formation of accretion-disk around the black hole. So, to avoid this complexity we assume that δu_ϕ can be neglected. The same argument also holds for the γ_ϕ. So, with this assumption $\delta T_{\phi R} = 0$. Therefore, the equations reduce to :

$$\delta_{\omega,q_2,q_3} R_{ij} = 0 \,. \tag{5.81}$$

We begin our calculation from these equations.

5.8 APPENDIX 3 : Determining some essential relations regarding the generalized McVittie metric

In the present work we need various relations between different quantities present in the line-element of the metric i.e. the metric-coefficients and transformation rules to shift from differentiation w.r.t. one coordinate system to the other. In this section, we are giving these relations and formula which have been used in our work in the present chapter.
The integrating factor F in the equation 5.7 satisfies the differential equation [192]:

$$\frac{\partial}{\partial R}\left(\frac{1}{F}\right) = \frac{\partial}{\partial t}\left(\frac{\beta}{F}\right), \tag{5.82}$$

where β is the quantity $\frac{C}{A^2 - C^2}$. To simplify this equation, first of all we express the partial derivative w.r.t. the radial coordinate $R(t,r)$ in Nolan-gauge, in terms of the partial derivative w.r.t. isotropic time coordinate t. As already stated the coordinate R is given by,

$$R = a(t) r \left(1 + \frac{M_H(t)}{2a(t)r}\right)^2 .$$

So, the partial derivative of R w.r.t. t is given by :

$$\frac{\partial R}{\partial t} = \dot{a}r\Big(1+\frac{M_H(t)}{2a(t)r}\Big)^2 + 2ar\Big(1+\frac{M_H(t)}{2a(t)r}\Big)\frac{1}{2r}\Big(\frac{1}{a}\dot{M}_H - \frac{M_H}{a^2}\dot{a}\Big)\,.$$

(It is quite clear that as the scale-factor a(t) and Hawking-Hayward quasi-local mass $M_H(t)$ are the functions of time(t) only, $\partial a/\partial t = da/dt \equiv \dot{a}$ and $\partial M_H/\partial t = dM_H/dt \equiv \dot{M}_H$.)
On simplifying the above expression of $\partial R/\partial t$, we obtain :

$$\frac{\partial R}{\partial t} = \left\{HR + M_H\Big(1+\frac{M_H}{2ar}\Big)\Big(\frac{\dot{M}_H}{M_H} - H\Big)\right\}\,. \tag{5.83}$$

As, $M_H(t) = m(t)a(t)$, it is easy to check that this can be written as :

$$\frac{\partial R}{\partial t} = \left\{HR + \dot{m}a\sqrt{\frac{\tilde{r}}{r}}\right\} = C(t,r)\,. \tag{5.84}$$

Therefore, we can say,

$$\frac{\partial}{\partial R} = \frac{1}{C(t,r)}\frac{\partial}{\partial t}\,. \tag{5.85}$$

Hence, the equation 5.82 can be written as :

$$\frac{1}{C}\frac{\partial}{\partial t}\Big(\frac{1}{F}\Big) = \frac{\partial}{\partial t}\Big(\frac{\beta}{F}\Big)\,, \tag{5.86}$$

$$\Rightarrow -F\frac{\partial \beta}{\partial F} = \frac{1}{C} - \beta\,. \tag{5.87}$$

Where,

$$\frac{1}{C} - \beta = \frac{1}{C} - \frac{C}{A^2 - C^2} = \frac{A^2 - 2C^2}{C(A^2 - C^2)}\,.$$

The above equation 5.87 has to be solved for getting the solution F.
Again, we have to determine the relation between $\frac{\partial}{\partial R}$ and $\frac{\partial}{\partial \bar{t}}$. We have already shown that

$$\frac{\partial R}{\partial t} = C(t,R)\,,$$

and the time-coordinate $\bar{t}$, we are working with, is given by :

$$d\bar{t} = \frac{1}{F}\Big(dt + \frac{C}{A^2 - C^2}dR\Big)\,. \tag{5.88}$$

From the above equation 5.88 we obtain :

$$\frac{\partial \bar{t}}{\partial R} = \frac{1}{F}\frac{\partial t}{\partial R} + \frac{C}{F(A^2 - C^2)}\,,$$

$$\Rightarrow \frac{\partial \bar{t}}{\partial R} = \frac{1}{FC\Big(1 - \frac{C^2}{A^2}\Big)}\,. \tag{5.89}$$

Hence, multiplying both sides with the differential operator $\frac{\partial}{\partial \bar{t}}$, we obtain :

$$\frac{\partial \bar{t}}{\partial R}\frac{\partial}{\partial \bar{t}} = \frac{\partial}{\partial R} = \frac{1}{FC\Big(1 - \frac{C^2}{A^2}\Big)}\frac{\partial}{\partial \bar{t}}\,.$$

Inserting the relation between $\frac{\partial}{\partial R}$ and $\frac{\partial}{\partial t}$, we see :

$$\frac{1}{C}\frac{\partial}{\partial t} = \frac{1}{FC\Big(1-\frac{C^2}{A^2}\Big)}\frac{\partial}{\partial \bar{t}}\,, \tag{5.90}$$

or,

$$\frac{\partial}{\partial t} = \frac{1}{F\Big(1-\frac{C^2}{A^2}\Big)}\frac{\partial}{\partial \bar{t}}\,. \tag{5.91}$$

Again, the partial derivative of R w.r.t. r gives :

$$\begin{aligned}\frac{\partial R}{\partial r} &= a(t)\Big(1+\frac{M_H(t)}{2a(t)r}\Big)^2 + 2a(t)r\Big(1+\frac{M_H(t)}{2a(t)r}\Big)\Big(-\frac{M_H(t)}{2a(t)r^2}\Big)\\ &= a(t)\Big(1+\frac{M_H(t)}{2a(t)r}\Big)\Big(1-\frac{M_H(t)}{2a(t)r}\Big)\\ &= a(t)\Big(1-\frac{M_H^2(t)}{(2a(t)r)^2}\Big)\,.\end{aligned} \tag{5.92}$$

So, the relation between the partial derivatives w.r.t. R and r can be written as :

$$\frac{\partial}{\partial r} = a(t)\Big(1-\frac{M_H^2(t)}{(2a(t)r)^2}\Big)\frac{\partial}{\partial R}\,. \tag{5.93}$$

5.9 APPENDIX 4 : Some relations regarding Fourier-transformation and Convolution theorem, which have been applied

It is to be noted that every term appearing in the integrands in the equations 5.35 and 5.36 can be written as a multiplication of an unperturbed term (containing unperturbed quantities appearing in the metric-coefficients) and a perturbation term. While some of the perturbation terms have partial derivatives w.r.t. t acting on the perturbations. Considering these two type of terms separately, we first write their Fourier-integral versions, inside the overall Fourier-integral from time-space to frequency-space. Thereafter, the partial-derivatives of t acting on the perturbations will give rise to a factor of $i\sigma''$, where σ'' is the integrating variable and consequently $(i\sigma'')^2$. We shall consider the perturbation term with those factors as a whole. Thereafter, we apply the 'Covolution theorem' for Fourier-transformations for the product of these two types of terms viz. the unperturbed term and the perturbation term. According to the 'Covolution theorem', one of these terms, say the perturbation term will be a function pf $(\sigma - \sigma')$ inside the integral, where σ is the integrating variable and the Fourier-transform is evaluated at the frequency σ'. We take the case of $\sigma' = 0$, which actually makes the Fourier-integral proportional to the mean of the integrand. Then, we equate the integrands from both sides of the equations.
We describe the procedure in a generalized way. Suppose, we have the equation :

$$f(r,\theta,t)\delta\alpha(r,\theta,t) = g(r,\theta,t)\delta\beta(r,\theta,t)\,, \tag{5.94}$$

where $f(r,\theta,t)$ and $g(r,\theta,t)$ consists of unperturbed parameters, present in the metric coefficients and functions of r , θ and t. $\delta\alpha$ and $\delta\beta$ are linear perturbations, which are all time(t)-varying . The aim is to map this equation from time-space to frequency-space employing Fourier-transformation. To map the time-dependence of the overall equation from time-space to frequency-space we fourier-transform both sides of the equation directly. This gives :

$$\int_0^\infty f(t)\delta\alpha(t)\, e^{-i\sigma' t} dt = \int_0^\infty g(t)\delta\beta(t)\, e^{-i\sigma' t} dt\,. \tag{5.95}$$

Now applying the 'Convolution theorem' for Fourier-transforms, we write the equation 5.88 as :

$$\int_{-\infty}^\infty f^\dagger(\sigma-\sigma')\delta\alpha^\dagger(\sigma) d\sigma = \int_{-\infty}^\infty g^\dagger(\sigma-\sigma')\delta\beta^\dagger(\sigma) d\sigma\,. \tag{5.96}$$

If we take the case of $\sigma'=0$ for the equation 5.88, then it gives :

$$\int_{-\infty}^\infty f^\dagger(\sigma)\delta\alpha^\dagger(\sigma) d\sigma = \int_{-\infty}^\infty g^\dagger(\sigma)\delta\beta^\dagger(\sigma) d\sigma\,. \tag{5.97}$$

Considering the case of $\sigma'=0$ physically means that the Fourier-integral becomes proportional to the mean of the integrand. Thereafter equating the integrands of the integrals from both sides of the equation 5.97 (as this can be done without any loss of generality), we get :

$$f^\dagger(\sigma)\delta\alpha^\dagger(\sigma) = g^\dagger(\sigma)\delta\beta^\dagger(\sigma)\,. \tag{5.98}$$

Again, we consider that the LHS of the equation 5.94 can also be written as $\tilde{f}\delta\tilde{\alpha}$ i.e. $\tilde{f}\delta\tilde{\alpha} = f\delta\alpha$. This is nothing but we define the linear perturbation in a different way. Then, the Fourier-transform of both $\tilde{f}\delta\tilde{\alpha}$ and $f\delta\alpha$ will be same viz. :

$$\int_0^\infty f(t)\delta\alpha(t)\, e^{-i\sigma' t} dt = \int_0^\infty \tilde{f}(t)\delta\tilde{\alpha}(t)\, e^{-i\sigma' t} dt\,. \tag{5.99}$$

Applying the Convolution theorem, similarly as we did for getting equation 5.96 from 5.95, and then taking the case of zero-frequency i.e. $\sigma'=0$, from the equation 5.99 we get :

$$\int_{-\infty}^\infty f^\dagger(\sigma)\delta\alpha^\dagger(\sigma) d\sigma = \int_{-\infty}^\infty \tilde{f}^\dagger(\sigma)\delta\tilde{\alpha}^\dagger(\sigma) d\sigma\,, \tag{5.100}$$

which in turn gives :

$$f^\dagger(\sigma)\delta\alpha^\dagger(\sigma) = \tilde{f}(\sigma)^\dagger\delta\tilde{\alpha}^\dagger(\sigma)\,, \tag{5.101}$$

by equating the integrands. The result of equation 5.101 is not only valid for the case of a term with perturbation, but also for any function i.e. say if $f = f_1(t)f_2(t) = \tilde{f}_1(t)\tilde{f}_2(t)$, then it implies $f_1^\dagger(t)f_2^\dagger(t) = \tilde{f}_1^\dagger(t)\tilde{f}_2^\dagger(t)$.

Now, we analyse the case where the functions $f(t)$ and $g(t)$ contains partial differential-operators w.r.t. t in general, as this is actually the case of perturbation equations in our work. We write $\hat{f}(t)$ as $\hat{f}(t) \equiv f(t) + h(t)\frac{\partial}{\partial t}$. So, the Fourier-transform of the part $f(t)\delta\alpha(t)$ proceeds in the usual way, as has been shown. The quantity $h(t)\frac{\partial}{\partial t}(\delta\alpha)$ is of our interest here. Its Fourier-transform gives :

$$\left\{\left(h(t)\frac{\partial}{\partial t}\right)\delta\alpha\right\}^\dagger = \frac{1}{2\pi}\int_0^\infty \left(h(t)\frac{\partial}{\partial t}\right)\delta\alpha(t)\, e^{-i\sigma' t} dt\,. \tag{5.102}$$

Substituting $h(t)$ and $\delta\alpha(t)$ in terms of their Fourier-transforms from time-space to frequency-space in the integrand on the RHS of the above equation 5.102, we get :

$$\begin{aligned}\left\{\left(h(t)\frac{\partial}{\partial t}\right)\delta\alpha\right\}^{\dagger} &= \left(\frac{1}{2\pi}\right)^{3}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty}h^{\dagger}(\sigma)e^{i\sigma t}d\sigma\right)\frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty}\delta\alpha^{\dagger}(\sigma'')e^{i\sigma'' t}d\sigma''\right)\, e^{-i\sigma' t}dt\,,\\ &= \left(\frac{1}{2\pi}\right)^{3}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty}h^{\dagger}(\sigma)e^{i\sigma t}d\sigma\right)\left(\int_{-\infty}^{\infty}i(\sigma''-\sigma')\delta\alpha^{\dagger}(\sigma'')e^{i\sigma'' t}d\sigma''\right)\, e^{-i\sigma' t}dt\,,\\ \Rightarrow \left\{\left(h(t)\frac{\partial}{\partial t}\right)\delta\alpha\right\}^{\dagger} &= \left(\frac{1}{2\pi}\right)^{3}\int_{0}^{\infty}\int_{-\infty}^{\infty}h^{\dagger}(\sigma)\int_{-\infty}^{\infty}i(\sigma''-\sigma')\delta\alpha^{\dagger}(\sigma'')\, e^{i(\sigma+\sigma''-\sigma')t}dt d\sigma d\sigma''\,.\end{aligned} \tag{5.103}$$

The RHS of the above equation 5.103 contains the integral of $e^{i(\sigma+\sigma''-\sigma')t}$ over time. This is just the Fourier-transform of a plane-wave, which is actually the Dirac-Delta function viz. :

$$\frac{1}{2\pi}\int_{0}^{\infty}e^{i(\sigma+\sigma''-\sigma')t}dt = \delta(\sigma+\sigma''-\sigma') = \delta(\sigma'-\sigma-\sigma'')\,. \tag{5.104}$$

[7] So, we write the equation 5.103 as :

$$\left\{\left(h(t)\frac{\partial}{\partial t}\right)\delta\alpha\right\}^{\dagger} = \left(\frac{1}{2\pi}\right)^{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h^{\dagger}(\sigma)\, i(\sigma''-\sigma')\delta\alpha^{\dagger}(\sigma'')\delta(\sigma'-\sigma-\sigma'')d\sigma d\sigma''\,. \tag{5.105}$$

Using the property of Dirac-Delta function on the RHS of the equation 5.105, integrating over σ'', we get :

$$\begin{aligned}\left\{\left(h(t)\frac{\partial}{\partial t}\right)\delta\alpha\right\}^{\dagger} &= \left(\frac{1}{2\pi}\right)^{2}\int_{-\infty}^{\infty}h^{\dagger}(\sigma)\, i(\sigma'-\sigma-\sigma')\delta\alpha^{\dagger}(\sigma'')d\sigma\\ &= \left(\frac{1}{2\pi}\right)^{2}\int_{-\infty}^{\infty}h^{\dagger}(\sigma)\,(-i\sigma)\delta\alpha^{\dagger}(\sigma'-\sigma)d\sigma\,.\end{aligned} \tag{5.106}$$

Now, if we replace σ with $-\sigma$ in the integral on RHS of the above equation, then we shall obtain :

$$\left\{\left(h(t)\frac{\partial}{\partial t}\right)\delta\alpha\right\}^{\dagger} = \left(\frac{1}{2\pi}\right)^{2}\int_{-\infty}^{\infty}h^{\dagger}(-\sigma)\,(+i\sigma)\delta\alpha^{\dagger}(\sigma'+\sigma)d\sigma\,. \tag{5.107}$$

Then for the case of $\sigma'=0$, this gives :

$$\left\{\left(h(t)\frac{\partial}{\partial t}\right)\delta\alpha\right\}^{\dagger} = \left(\frac{1}{2\pi}\right)^{2}\int_{-\infty}^{\infty}h^{\dagger}(-\sigma)\,(+i\sigma)\delta\alpha^{\dagger}(\sigma)d\sigma\,. \tag{5.108}$$

[7] The last step uses the property that Dirac-Delta function is symmetric.

5.10 APPENDIX 5 : Analyzing the quantities $\frac{(\mathbb{F}(r,t)+i\sigma)}{(\mathscr{F}(r,t)+i\sigma)}$, $\frac{\mathbb{F}(r,t)}{\mathscr{F}(r,t)}$ and the approximations applicable to these

5.10.1 Expressing the ratio $\frac{\mathbb{F}(r,t)}{\mathscr{F}(r,t)}$ conveniently in terms of C, A, Δ , F and their derivatives :

In this sub-section, we are going to express the quantities $\mathbb{F}$ and $\mathscr{F}$ in terms of C, A, Δ , F and their derivatives ; and then in the next sub-section we shall investigate some approximations, which will be applicable to our calculations.
The quantity $\frac{\mathbb{F}(r,t)}{\mathscr{F}(r,t)}$ can be expressed as :

$$\frac{\mathbb{F}(r,t)}{\mathscr{F}(r,t)} = \frac{\frac{\Delta}{R^4}\left(1-\frac{C^2}{A^2}\right)F\frac{\partial}{\partial \bar{t}}\left\{\frac{\Delta}{R^4}\left(1-\frac{C^2}{A^2}\right)F\right\}^{-1}}{\frac{F}{R^4}\frac{\partial}{\partial \bar{t}}\left(\frac{R^4}{F}\right)}, \tag{5.109}$$

$$\Rightarrow \frac{\mathbb{F}(r,t)}{\mathscr{F}(r,t)} = 1 + \frac{\Delta\left(1-\frac{C^2}{A^2}\right)\frac{\partial}{\partial \bar{t}}\left(\Delta\left(1-\frac{C^2}{A^2}\right)\right)^{-1}}{\frac{F}{R^4}\frac{\partial}{\partial \bar{t}}\left(\frac{R^4}{F}\right)}. \tag{5.110}$$

The denominator in the additional term with 1 on the RHS of the above equation 5.110 can be written as :

$$\frac{1}{\frac{F}{R^4}\frac{\partial}{\partial \bar{t}}\left(\frac{R^4}{F}\right)} = \left\{F\frac{\partial}{\partial \bar{t}}\left(\frac{1}{F}\right) + \frac{1}{R^4}\frac{\partial}{\partial \bar{t}}R^4\right\}^{-1}. \tag{5.111}$$

Using the differential relation satisfied by F given in 5.82 we can obtain the following relation :

$$F\left\{\frac{\partial}{\partial \bar{t}}\left(\frac{1}{F}\right)\right\} = \left(\frac{1}{C}-\beta\right)^{-1}\frac{\partial \beta}{\partial \bar{t}}, \tag{5.112}$$

where the quantity β is given by : $\beta = \frac{C}{A^2-C^2}$; and using the relations between $\frac{\partial}{\partial R}$ and $\frac{\partial}{\partial \bar{t}}$, derived in the appendix 3 i.e. section 5.8, we can easily get :

$$\frac{1}{R^4}\frac{\partial}{\partial \bar{t}}R^4 = \frac{4}{R}FC\left(1-\frac{C^2}{A^2}\right). \tag{5.113}$$

We write the detailed expressions of $\frac{\partial \beta}{\partial \bar{t}}$ and $\frac{\partial A}{\partial \bar{t}}$ respectively as :

$$\frac{\partial \beta}{\partial \bar{t}} = \frac{1}{(A^2-C^2)^2}\left\{(A^2+C^2)\frac{\partial C}{\partial \bar{t}} - 2CA\frac{\partial A}{\partial \bar{t}}\right\}. \tag{5.114}$$

and

$$\frac{\partial A}{\partial \bar{t}} = \frac{\partial}{\partial \bar{t}}\left(1-\frac{2M_H}{R}\right) = -2\left\{-\frac{M_H}{R^2}\frac{\partial R}{\partial \bar{t}} + \frac{1}{R}\frac{\partial M_H}{\partial \bar{t}}\right\}. \tag{5.115}$$

Here, $\frac{\partial M_H}{\partial \bar{t}}$ can be expressed as :

$$\frac{\partial M_H}{\partial \bar{t}} = F\left(1-\frac{C^2}{A^2}\right)\frac{\partial M_H}{\partial t} = M_H F\left(1-\frac{C^2}{A^2}\right)\left(H+\frac{\dot{m}}{m}\right).$$

Substituting the expression of the $\frac{\partial R}{\partial \bar{t}}$ in the equation 5.115, we obtain :

$$\frac{\partial A}{\partial \bar{t}} = -2\frac{M_H F}{R^2}\left(1 - \frac{C^2}{A^2}\right)\left\{-C + HR + R\frac{\dot{m}}{m}\right\}. \tag{5.116}$$

As, $C = HR + \dot{m}a\sqrt{\frac{\tilde{r}}{r}}$, substituting it in the RHS of the above equation 5.116 we obatin :

$$\frac{\partial A}{\partial \bar{t}} = -2\frac{M_H F}{R^2}\dot{m}a\left(1 - \frac{C^2}{A^2}\right)\left\{-\sqrt{\frac{\tilde{r}}{r}} + \frac{\tilde{r}}{m}\right\}. \tag{5.117}$$

Substituting the expressions :

$$\sqrt{\frac{\tilde{r}}{r}} = \left(1 + \frac{M_H}{2ra}\right) = \left(1 + \frac{m}{2r}\right)$$

and

$$\frac{\tilde{r}}{m} = \frac{r}{m}\left(1 + \frac{m}{2r}\right)^2 = \left(\frac{r}{m} + 1 + \frac{m}{4r}\right),$$

on the RHS of the equation 5.117, we get :

$$\frac{\partial A}{\partial \bar{t}} = -2\frac{M_H F}{R^2}\dot{m}a\left(1 - \frac{C^2}{A^2}\right)\left(\frac{r}{m} - \frac{m}{4r}\right). \tag{5.118}$$

Calculating the detailed expression of the quantity $\frac{\partial C}{\partial \bar{t}}$ we obtain :

$$\frac{\partial C}{\partial \bar{t}} = F\left(1 - \frac{C^2}{A^2}\right)\left\{-H^2R + 2\dot{m}\dot{a}\left(1 + \frac{m}{2r}\right) + \frac{\dot{m}^2 a}{2r} + \ddot{m}a\left(1 + \frac{m}{2r}\right)\right\}. \tag{5.119}$$

Hence, the denominator in the additive term with 1 on the RHS of the equation 5.110 can be written as :

$$\frac{F}{R^4}\frac{\partial}{\partial \bar{t}}\left(\frac{R^4}{F}\right) = \left(\frac{1}{C} - \beta\right)^{-1}\frac{\partial \beta}{\partial \bar{t}} + \frac{4}{R}FC\left(1 - \frac{C^2}{A^2}\right) = C\left(1 - \frac{C^2}{A^2}\right)\left\{\left(1 - \frac{2C^2}{A^2}\right)^{-1}\frac{\partial \beta}{\partial \bar{t}} + \frac{4}{R}F\right\}.$$

While the numerator of that term is given by :

$$\Delta\left(1 - \frac{C^2}{A^2}\right)\frac{\partial}{\partial \bar{t}}\left(\Delta\left(1 - \frac{C^2}{A^2}\right)\right)^{-1}.$$

Hence, the additive term with 1 on the RHS of the equation 5.110 can be written as :

$$\frac{\Delta\left(1 - \frac{C^2}{A^2}\right)\frac{\partial}{\partial \bar{t}}\left(\Delta\left(1 - \frac{C^2}{A^2}\right)\right)^{-1}}{\frac{F}{R^4}\frac{\partial}{\partial \bar{t}}\left(\frac{R^4}{F}\right)} = -\frac{\left(1 - \frac{C^2}{A^2}\right)^{-2}}{\Delta C}\frac{\frac{\partial}{\partial \bar{t}}\left(\Delta\left(1 - \frac{C^2}{A^2}\right)\right)^{-1}}{\left\{\left(1 - \frac{2C^2}{A^2}\right)^{-1}\frac{\partial \beta}{\partial \bar{t}} + \frac{4}{R}F\right\}}. \tag{5.120}$$

Writing the detailed expressions of numerator and denominator in the additive term with 1 on the RHS of the equation 5.110, in terms of C, A, Δ, F and their derivatives w.r.t. $\bar{t}$, we obtain :

$$\frac{\Delta\left(1 - \frac{C^2}{A^2}\right)\frac{\partial}{\partial \bar{t}}\left(\Delta\left(1 - \frac{C^2}{A^2}\right)\right)^{-1}}{\frac{F}{R^4}\frac{\partial}{\partial \bar{t}}\left(\frac{R^4}{F}\right)} = \frac{\left\{\left(\frac{-2}{A^2}\frac{\partial C}{\partial \bar{t}} + \frac{2C}{A^3}\frac{\partial A}{\partial \bar{t}}\right) + \left(1 - \frac{C^2}{A^2}\right)\frac{1}{\Delta}\frac{\partial \Delta}{\partial \bar{t}}\right\}}{\left(1 - \frac{2C^2}{A^2}\right)^{-1}\left\{\left(1 + \frac{C^2}{A^2}\right)\frac{1}{A^2}\frac{\partial C}{\partial \bar{t}} - \frac{2C}{A^3}\frac{\partial A}{\partial \bar{t}}\right\} + \left(1 - \frac{C^2}{A^2}\right)^2\frac{4}{R}F}. \tag{5.121}$$

5.10.2 Checking the order of different quantities present in the ratio $\frac{\mathbb{F}(r,t)}{\mathscr{F}(r,t)}$ and applying approximation :

If we examine the ratio given in equation 5.121 in the previous sub-section, then in the numerator and denominator of the ratio on the RHS of the equation 5.121, the quantities $\left(\frac{-2}{A^2}\frac{\partial C}{\partial t}+\frac{2C}{A^3}\frac{\partial A}{\partial t}\right)$ and $\left(1-\frac{2C^2}{A^2}\right)^{-1}\left\{\left(1+\frac{C^2}{A^2}\right)\frac{1}{A^2}\frac{\partial C}{\partial t}-\frac{2C}{A^3}\frac{\partial A}{\partial t}\right\}$ should have same order of magnitude within finite distance from the black hole [8]. Therefore, the order of the magnitude of the ratio given in equation 5.121 would depend mainly on the quantities $\frac{1}{\Delta}\frac{\partial \Delta}{\partial t}$ and F/R, if their order of magnitude is higher than the former quantities.

Therefore, now we have to check the relative significance of different quantities in the ratio given in equation 5.121 for having an idea on its overall order of magnitude.

First of all, we check the relative significance of the quantities : $\left(1-\frac{C^2}{A^2}\right)\frac{1}{\Delta}\frac{\partial \Delta}{\partial t}$ and $\left(1-\frac{C^2}{A^2}\right)^2\frac{4F}{R}$. Their ratio can be expressed as :

$$\frac{\left(1-\dfrac{C^2}{A^2}\right)\dfrac{1}{\Delta}\dfrac{\partial\Delta}{\partial t}}{\left(1-\dfrac{C^2}{A^2}\right)^2\dfrac{4F}{R}}=\frac{\dfrac{2C(R-M_H)-2\dot{M}_H R}{R^2-2M_H R}}{\dfrac{4}{R}}=\frac{1}{2}\left\{C\frac{\left(1-\frac{M_H}{R}\right)}{\left(1-2\frac{M_H}{R}\right)}-\frac{\dot{M}_H}{\left(1-2\frac{M_H}{R}\right)}\right\}. \tag{5.122}$$

Hence, if magnitude of $C << 1$ and magnitude of $\dot{M}_H << 1$, then the magnitude of the above ratio is also $<< 1$.

To establish the fact that the order of magnitude of C and $\dot{M}_H$ are $<< 1$, we plot these w.r.t. time from $10^{-25}\,s$ to $100\,s$ after Big-bang. We separately plot the two parts of C viz. HR/c and $\frac{G}{c^3}\dot{m}a\sqrt{\frac{\tilde{r}}{r}}$; as these vary with time in different ways.

It is to be noted that the radial-distance R is arbitrary in this case, but that does not imply that it can be taken to theoretically-infinite or asymptotic distance. This is because at theoretically-infinite distance the generalized McVittie metric would reduce to FLRW-metric. The evolution of perturbations, around the PBH, is mainly to be studied within finite distance from the PBH. To set a characteristic length-scale for plotting C w.r.t. time, we use the comoving Schwarzschild length-scale $R_s=\frac{2GM_H}{c^2}$ (not the horizon), for the PBHs of maximum mass available at a certain instant of time in early Universe, which is just the horizon-mass m_h at that time. The horizon-mass m_h at a time t seconds after Big-bang is given by $m_h = 10^{35}t\,kg$ [22, 24]. In this way, we depict the maximum possible value of the comoving Schwarzschild length-scale R_s at that time. The plot of HR_s/c w.r.t. time has been shown in the figure 5.1.

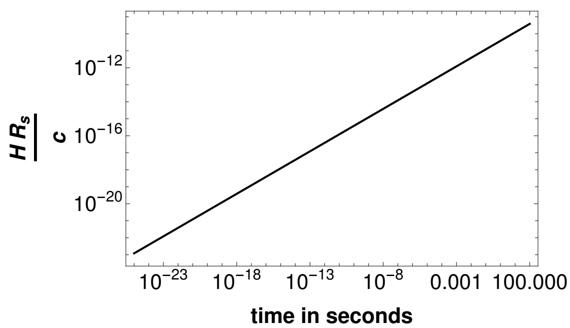

Figure 5.1: Plot in log-log scale, showing the variation of $\frac{HR_s}{c}$ with time from $10^{-25}\,s$ to $100\,s$ after Big-bang.

[8] It is quite clear that the quantities $\left(1-\frac{2C^2}{A^2}\right)$, $\left(1+\frac{C^2}{A^2}\right)$ and $\left(1-\frac{C^2}{A^2}\right)\sim 1$ within finite distance from the primordial black hole described by the generalized McVittie metric.

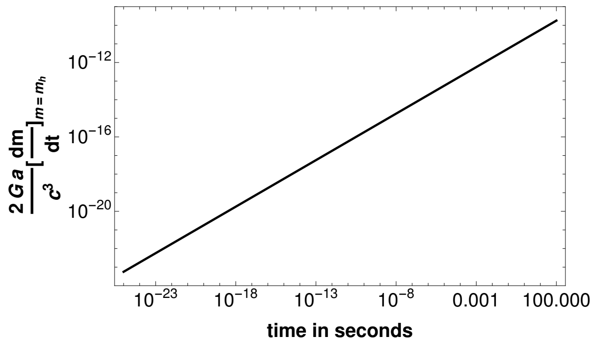

Figure 5.2: Plot in log-log scale, showing the variation of $\frac{2G}{c^3}a[\dot{m}]_{m=m_h}$ with time from $10^{-25}\,s$ to $100\,s$ after Big-bang.

On the other-hand, for $\frac{G}{c^3}\dot{m}a\sqrt{\frac{\tilde{r}}{r}}$, the quantity $\sqrt{\frac{\tilde{r}}{r}}$ for the Schwarzschild length-scale would be 2 ; as for the Schwarzschild radius, isotropic radial-coordinate $r = \frac{Gm}{2c^2}$. Again, we use $\dot{m}$ at $m = m_h$, thereby taking the maximum possible value of PBH-mass at a certain instant of time. The plot of $\frac{2G}{c^3}a[\dot{m}]_{m=m_h}$ w.r.t. time has been shown in the figure 5.2. In this case, it is worth mentioning that here we are considering the mass-range of PBHs such that their mass-change due to Hawking-evaporation would be negligible and the only significant way of mass-change is the spherical accretion of the surrounding radiation.

Another issue is to be noted in these cases, that we have shown these plots for time till 100 s after Big-bang, mainly because this is the order of time (at which Big-bang nucleosynthesis occurred), at which new PBH-production, by direct gravitational-collapse of sufficiently deep density-perturbations, is predicted to be stopped.

Next, we show the plot of $\frac{G}{c^3}[\dot{M}_H]_{m=m_h}$, which appears in the second term of the expression, on the RHS of the equation 5.122, w.r.t. time in the figure 5.3. [9]

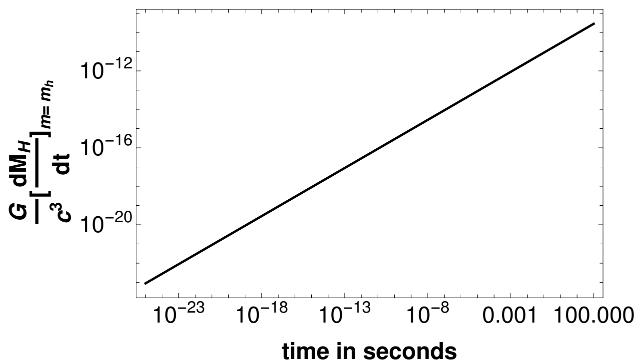

Figure 5.3: Plot in log-log scale, showing the variation of $\frac{G}{c^3}[\dot{M}_H]_{m=m_h}$ with time from $10^{-25}\,s$ to $100\,s$ after Big-bang.

So, we see that all the three quantities $\frac{HR_s}{c}$, $\frac{2G}{c^3}a[\dot{m}]_{m=m_h}$ and $\frac{G}{c^3}[\dot{M}_H]_{m=m_h}$ have magnitudes within order of 10^{-23} to 10^{-9}, for the range of time from $10^{-25}\,s$ to $100\,s$ after Big-bang. [10]
Therefore, from these plots, it is clear that the approximation based on $C << 1$ and $\dot{M}_H << 1$ are very well valid in the specified range of time.

[9] One part in $\frac{G}{c^3}[\dot{M}_H]_{m=m_h}$ viz. the $\frac{G}{c^3}a[\dot{m}]_{m=m_h}$, has already been plotted in figure 5.2, with the factor 2. So, if the other part $\frac{G}{c^3}\dot{a}m_h$ is less than or equal to the former, then the order of the whole $\frac{G}{c^3}[\dot{M}_H]_{m=m_h}$ would be same as that of $\frac{G}{c^3}a[\dot{m}]_{m=m_h}$; and in fact this is happening in this case.

[10] Another issue is also to be noted that the overall order of the ratio given in equation 5.122 may be lesser (even before $10^{-23}\,s$) as in the expression derived in equation 5.122, the term containing $\dot{M}_H$ is subtracted from the term containing C.

We now check the relative significance of the quantities : $\left(1-\frac{C^2}{A^2}\right)\frac{1}{\Delta}\frac{\partial\Delta}{\partial\tilde{t}}$ and $\left(\frac{-2}{A^2}\frac{\partial C}{\partial\tilde{t}}+\frac{2C}{A^3}\frac{\partial A}{\partial\tilde{t}}\right)$. We take their ratio :

$$\frac{\left(\frac{-2}{A^2}\frac{\partial C}{\partial\tilde{t}}+\frac{2C}{A^3}\frac{\partial A}{\partial\tilde{t}}\right)}{\left(1-\frac{C^2}{A^2}\right)\frac{1}{\Delta}\frac{\partial\Delta}{\partial\tilde{t}}}=\frac{2}{A^2}\left[\frac{-\left\{H^2R+\left(1+\frac{m}{2r}\right)(2\dot{m}\dot{a}+\ddot{m}a)+\frac{\dot{m}^2a}{2r}\right\}+\frac{C}{A}\left\{-2\frac{M_H}{R^2}\dot{m}a\left(1+\frac{m}{2r}\right)\left(\frac{r}{m}-\frac{1}{2}\right)\right\}}{\frac{1}{R}\left\{C\frac{\left(1-\frac{M_H}{R}\right)}{\left(1-2\frac{M_H}{R}\right)}-\frac{\dot{M}_H}{\left(1-2\frac{M_H}{R}\right)}\right\}}\right]. \tag{5.123}$$

Now, it is to be noted that in the ratio given in the above equation 5.123, the quantities containing $\dot{m}$, $\ddot{m}$ or $\dot{M}_H$, have the constant Gc^{-3} with each of them, as we have described in the APPENDIX-1 viz. section 5.6. Furthermore, these quantities have a or $\dot{a}$. Hence, all of these quantities would be of very much smaller order in the scenario of our interest, where the radial distance from the PBH is finite. The dominant term in the numerator of the ratio given in equation 5.123 would be H^2R. Therefore, the overall ratio would have the order of $\sim -\frac{2}{A^2}\frac{H^2R}{H}=-\frac{2}{A^2}HR$ or $\frac{2}{A^2}\dot{a}\tilde{r}$. As we are interested in the case where the radial distance from the PBH is finite, $A\sim 1$ and in the early radiation dominated era,[11] $\dot{a}<<1$, the resultant order of the ratio in equation 5.123 is very smaller than 1. This implies that, in the scenario of our interest the quantity $\left(\frac{-2}{A^2}\frac{\partial C}{\partial\tilde{t}}+\frac{2C}{A^3}\frac{\partial A}{\partial\tilde{t}}\right)$ may be neglected with respect to the quantity $\left(1-\frac{C^2}{A^2}\right)\frac{1}{\Delta}\frac{\partial\Delta}{\partial\tilde{t}}$. Again, we have previously shown that the quantity $\left(1-\frac{C^2}{A^2}\right)\frac{1}{\Delta}\frac{\partial\Delta}{\partial\tilde{t}}$ is negligible in comparison with the quantity $\left(1-\frac{C^2}{A^2}\right)^2\frac{4F}{R}$. It is quite clear that the quantity $\left(1-\frac{2C^2}{A^2}\right)^{-1}\left\{\left(1+\frac{C^2}{A^2}\right)\frac{1}{A^2}\frac{\partial C}{\partial\tilde{t}}-\frac{2C}{A^3}\frac{\partial A}{\partial\tilde{t}}\right\}\sim\left(\frac{-2}{A^2}\frac{\partial C}{\partial\tilde{t}}+\frac{2C}{A^3}\frac{\partial A}{\partial\tilde{t}}\right)$. Hence, the ratio in the equation 5.121 is of order $<<1$. Thus, this analysis implies that the additive term in the equation 5.110 with 1 can be neglected making the ratio $\mathbb{F}/\mathscr{F}\approx 1$ or, $\mathbb{F}\approx\mathscr{F}$. Again, in the quantity $\frac{\mathbb{F}+i\sigma}{\mathscr{F}+i\sigma}$, if the frequency of the mode is not too small (i.e. if we neglect the ultra-low frequency modes), then too, the quantity $\frac{\mathbb{F}+i\sigma}{\mathscr{F}+i\sigma}$ would be ≈ 1.

[11] the only case where $\dot{a}$ may be near order 1 is the time just after the end of inflation

Chapter 6

Summary of the thesis and future prospects

In this thesis we have inspected some phenomena of gravitational waves associated with primordial black holes (PBHs), which have the prospect to distinctly distinguish their properties theoretically and observationally. Three of these phenomena are mainly associated with PBHs of early Universe. Hence, exploration of these phenomena should increase our insight about the early Universe of concerned times. The works included in this thesis have some future prospects for further exploration of these phenomena with more precision or some other aspects related to these.

In the second chapter, we investigate the stochastic gravitational wave background produced by PBH binaries during their early inspiral stage while accreting high-density radiation surrounding those PBHs in the early universe. We first show that the gravitational wave amplitude produced from a PBH binary has correction terms because of the rapid rate of increase in masses of the PBHs. These correction terms arise due to non-vanishing first and second time-derivatives of the PBH-masses and their contribution to the overall second-order time derivative of quadrupole-moment tensor. We find that some of these correction terms are not only significant in comparison with the main term but may be even dominant over the main term for certain ranges of time in the early Universe. The significance of these correction terms persists for the overall stochastic gravitational wave background produced from the PBH binaries. We have shown that the spectral density of the stochastic gravitational wave background, produced from such accreting PBH binaries, lie within the detectability-range of some present and future gravitational wave detectors.

In context of the above topic, it will be of preliminary importance to evaluate the gravitational wave amplitude generated from the merging stage of the PBH binaries while accreting the surrounding high-density radiation in the early Universe and subsequently to calculate the relevant parameters of the stochastic gravitational wave background. This will be interesting to investigate whether the correction terms in the corresponding stochastic gravitational wave background will be significant in comparison to the main term or not, and if those are found to be significant, then what will be the extent of the significance. However, it is almost evident that accurate evaluation in this scenario will need the implementation of numerical general relativistic techniques, as finding the merging stage dynamics and the gravitational wave signal emitted from binary of black holes in the merging stage can hardly be done analytically (unless the ratio of the masses of the black holes are very high). In fact, as it is expected that the gravitational wave signal emitted from any binary of black holes in the merging stage has the highest amplitude in comparison to the early and late inspiral stages and the ringdown stage, the contribution of the merging stages of the concerned PBH binaries are expected to enhance the spectral density of the stochastic background. The detectability of the stochastic background is also expected to get enhanced if we consider gravitational waves from merging stages of the PBH binaries along with those of the early and late inspiral stages.

In the third chapter, we consider the equation of motion of a charged particle or a charged compact object in curved space-time, under the reaction of electromagnetic radiation and also consider a physical situation such that the charged particle or compact object emits gravitational radiation, thereby gravitational radiation reaction also acts on it. We investigate the effect of this metric perturbation i.e. the gravitational radiation on the electromagnetic self-force. We show that, besides the interaction terms derived by P. Zimmerman and E. Poisson [98], additional perturbative terms are generated, due to perturbation of the electromagnetic self-force by the metric perturbation. We discuss the conditions of significance of these perturbative terms and also the interaction terms with respect to the gravitational self-force in various astrophysical and cosmological cases. We find that, in some astrophysical and cosmological phenomena, these perturbative terms can have significant effect in comparison with the gravitational radiation-reaction term.

The investigation, mentioned above, motivates us to examine the perturbation of the 'electromagnetic tail-term' due to the metric fluctuation i.e. the gravitational radiation emitted from the charged particle or smaller compact object moving around a larger compact object. Then, the conditions of significance of those perturbative terms may be investigated.

In the fourth chapter, we study the change of masses of black holes due to spherical accretion of k-essence dilatonic ghost condensate model of dark energy and the impact of this change of masses on the evolution of the binaries formed by the black holes. We check the evolution of the eccentricity of orbits of such binaries, assuming the orbit to be highly elliptical during its formation and then after circularization of the orbits, we investigate the evolution of the radii of these orbits, while the masses are changing. Then, we compute the average power of the gravitational waves emitted from the binaries and compare it with the case when the masses of the black holes are constant. We find that the average power of the emitted gravitational wave increases significantly faster than the case when the masses are constant. Furthermore, we calculate the coalescence-time intervals of such binaries when the masses are changing. Comparing it with the case of constant masses, we estimate the reduction in coalescence-time intervals of the binaries due to change of the black hole masses. Our work described in this chapter signifies the effect of accretion of similar scalar field dark energies on the orbital evolution of binaries of black holes of certain mass-ranges, their coalescence-time-scale and as a consequence their merging rates too. We believe this will provide a different approach to observationally distinguish such models of dark energy, specially from the cosmological constant Λ, in the emerging era of gravitational wave astronomy.

The above analysis on the effect of continuous accretion of dark energy on the change of masses of black holes and on modification of parameters of a binary consisting of such black holes can be done similarly for some other scalar field models of dark energy too besides the chosen one. We believe that following the similar style, several other scalar field dark energy models, which are not ruled out by traditional cosmological observations, can be constrained.

This also gives us another intriguing aspect of the phenomenon of accretion of scalar field dark energy by black holes in binary formations. As the black holes in binary formations move at relativistic speeds ($\sim c$) in the merging or coalescence-stage of a binary, it is then highly possible to produce shock waves of sound through the k-essence DGC dark energy or any similar scalar field dark energy, whose sound speed is then lesser ($\sim 0.1c$ in the present case) than the relativistic speeds of the black holes in the merging stage. If dark energy is indeed a sort of similar scalar field, then that sound should travel to us through it (as the present era is dominated by dark energy) and we would probably be able to detect that sound. This is another prospect of distinguishing such scalar field dark energy from the cosmological constant Λ, which may be tried to explore in future.

In the fifth chapter, we derive the equation governing the axial-perturbations in the space-time of a non-rotating uncharged primordial black hole (PBH), produced in early Universe, whose metric is taken as the generalized McVittie metric. The generalized McVittie metric is a cosmological black hole metric, proposed by V. Faraoni and A. Jacques in 2007 [190]. This describes the space-time of a Schwarzschild black hole embedded in FLRW-Universe, while allowing its mass-change. Our derivation has basic similarities with the procedure of derivation of S. Chandrasekhar, for deriving the Regge-Wheeler equation for Schwarzschild metric [191] ; but it has some distinct differences with that due to the complexity and time-dependency of the generalized McVittie metric. We show that after applying some approximations which are very well valid in the early radiation-dominated Universe, the overall equation governing the axial perturbations can be separated into radial and angular parts, among which the radial part is the intended one, as the angular part is identical to the case of Schwarzschild metric as expected. We identify the potential from the Schrödinger-like format of the equation and draw some physical interpretation from it.

The derivation of the equation governing axial perturbations in the generalized McVittie metric, its simplification and transformation in the Schrödinger-like form can be used to check whether there exists instabilities for the axial perturbations or not. Furthermore, this equation forms the basis for finding out the characteristic modes of vibration of a non-rotating uncharged PBH, whose space-time can be described by the generalized McVittie metric.

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